Boundary problems for the Ginzburg-Landau equation.

David Chiron

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David CHIRON

Laboratoire Jacques-Louis LIONS,
Université Pierre et Marie Curie Paris VI,
4, place Jussieu BC 187, 75252 Paris, France
E-Mail : chiron@ann.jussieu.fr

Abstract

We provide a study at the boundary for a class of equation including the Ginzburg-Landau equation as well as the equation of travelling waves for the Gross-Pitaevskii model. We prove Clearing-Out results and an orthogonal anchoring condition of the vortex on the boundary for the Ginzburg-Landau equation with magnetic field.

1 Introduction

This paper is devoted to the study at the boundary for the equation for the complex-valued function u in a bounded regular domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$,

$$i|\log \varepsilon|\vec{c}(x) \cdot \nabla u = \Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2) - |\log \varepsilon|^2 d(x) u, \tag{1}$$

where $\vec{c} : \Omega \to \mathbb{R}^N$ is a bounded lipschitz vector field, $d : \Omega \to \mathbb{R}_+$ is a lipschitz non negative bounded function and $\varepsilon > 0$ is a small parameter. For instance, the Ginzburg-Landau equation with magnetic field

$$(\nabla - i\vec{A}/2)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2) \tag{2}$$

is of the type considered. Another problem that can be written like equation (1) is the equation for the travelling waves for the Gross-Pitaevskii equation. This equation writes

$$i\frac{\partial \psi}{\partial t} + \Delta \psi + \psi(1 - |\psi|^2) = 0, \tag{3}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$. Travelling waves solutions to this equation are solutions of the form (possibly rotating the axis)

$$\psi(t, x) = U(x_1 - Ct, x_2, \ldots, x_N).$$

Equation (3) reads now on U

$$iC\frac{\partial U}{\partial x_1} = \Delta U + U(1 - |U|^2).$$

In dimension $N \geq 3$, if the propagation speed is small, it is convenient to perform the scaling

$$u(x) := U\Big(\frac{x}{\varepsilon}\Big), \qquad c := \frac{C}{\varepsilon|\log\varepsilon|}$$

(in dimension $N = 2$, the scaling for the speed is $C = \varepsilon$), and the equation becomes then

$$ic|\log\varepsilon|\frac{\partial u}{\partial x_1} = \Delta u + \frac{1}{\varepsilon^2}u(1 - |u|^2)$$

and we expect c to be of order one. This equation is of the type (1) with $d \equiv 0$ and $\vec{c} = c\vec{e}_1$. If $N = 2$, the equation is

$$i\frac{\partial u}{\partial x_1} = \Delta u + \frac{1}{\varepsilon^2}u(1 - |u|^2),$$

which is also of the considered type with $d \equiv 0$ and $\vec{c} = \frac{\vec{e}_1}{|\log\varepsilon|}$.

We will be interested in (1) in the asymptotic $\varepsilon \to 0$ with

$$\operatorname{div}\vec{c} = 0, \tag{4}$$

and we supplement this equation with

either the Dirichlet condition

$$u = g_\varepsilon \qquad \text{on } \partial\Omega, \tag{5}$$

either the Coulomb gauge and the homogeneous Neumann condition

$$\frac{\partial u}{\partial n} = 0 \qquad \text{and} \qquad \vec{c}\cdot n = 0 \qquad \text{on } \partial\Omega. \tag{6}$$

Furthermore, we will assume that there exists a constant $\Lambda_0 > 0$ independent of ε such that

$$|\vec{c}|^2_{L^\infty(\Omega)} + |\nabla\vec{c}|^2_{L^\infty(\Omega)} + |d|^2_{L^\infty(\Omega)} + |\nabla d|^2_{L^\infty(\Omega)} \leq \Lambda_0^2. \tag{7}$$

Finally, we may assume $0 < \varepsilon \leq \varepsilon_0(\Lambda_0) \leq 1/2$ small enough so that

$$\Lambda_0\varepsilon^{1/2}|\log\varepsilon|^2 \leq \frac{1}{2}. \tag{8}$$

To this problem, is associated the energy

$$E_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(a_\varepsilon(x) - |u|^2)^2}{2\varepsilon^2} = \int_\Omega e_\varepsilon(u),$$

where

$$a_\varepsilon(x) := 1 - d(x)\varepsilon^2|\log\varepsilon|^2.$$

1.1 Anchoring condition at the boundary

Our first result is about the anchoring condition of the vortex on the boundary for the Ginzburg-Landau equation with Neumann condition. Assuming the upper bound

$$E_\varepsilon(u) \le M|\log\varepsilon|,$$

for the function u, we expect that the energy of u concentrates at its vortices, which are curves Γ in dimension $N=3$. We therefore introduce the measure

$$\mu_\varepsilon := \frac{e_\varepsilon(u)}{|\log\varepsilon|}\,dx,$$

the mass of which is bounded by M by hypothesis. We may then assume, up to a subsequence, that as $\varepsilon \to 0$,

$$\mu_\varepsilon \rightharpoonup \mu_* \qquad \text{weakly as measures.}$$

Moreover, we define the $N-2$-dimensional density of μ_*

$$\Theta_*(x) := \liminf_{r\to 0} \frac{\mu_*(B_r(x))}{r^{N-2}}$$

and the geometrical support of μ_*

$$\Sigma_{\mu_*} := \{x \in \Omega,\ \Theta_*(x) > 0\}.$$

From Theorem 3 in [BOS], we know that Σ_{μ_*} is closed in Ω and countably $(N-2)$-rectifiable. Let us assume that the magnetic field $H = |\log\varepsilon|\,\mathrm{curl}\,\vec{c}$ obeys the London equation

$$-\Delta H + H = 2\pi\vec{\delta}_\Gamma.$$

We may then describe further Σ_{μ_*} near the boundary. In this regime of energy, Γ consists in a finite number of curves of finite length. Therefore, from London equation, we expect H to be of order one, that is

$$|\log\varepsilon|\cdot|\mathrm{curl}\,\vec{c}| = |H| \simeq 1,$$

and thus, since $\vec{c}\cdot n = 0$ on $\partial\Omega$,

$$|\vec{c}| \to 0 \qquad \text{if } \varepsilon \to 0.$$

Our result is concerned with the anchoring of Σ_{μ_*} at the boundary, under the only hypothesis

$$\vec{c}_\varepsilon \to 0 \qquad \text{in } \mathcal{C}^0(\bar{\Omega}) \ \text{ as } \varepsilon \to 0. \tag{9}$$

We note that by hypothesis, $\vec{c}_\varepsilon$ is bounded in $\mathcal{C}^{0,1}(\bar{\Omega})$, thus we may assume for a subsequence that $\vec{c}_\varepsilon \to \vec{c}$ in $\mathcal{C}^0(\bar{\Omega})$. We then only assume $\vec{c} = 0$.

In the case of the Neumann boundary condition (6), we will use the reflection principle. There exists $\delta > 0$ such that the nearest point projection map

$$\Pi : (\partial\Omega)_\delta \to \partial\Omega$$

is well-defined in the δ-neighborhood $(\partial\Omega)_\delta$ of $\partial\Omega$ and a smooth fibration. A point $x \in (\partial\Omega)_\delta$ may therefore be described by the couple (y,t), where $y = \Pi(x)$ is its projection on $\partial\Omega$ and

$t = \pm dist(x, \partial\Omega) = \pm\|x - \Pi(x)\|$, the sign $\pm$ being $+$ if x is inside Ω and $-$ otherwise. We then define the reflection map

$$\phi : W := \bar{\Omega}^c \cap (\partial\Omega)_\delta \to V := \Omega \cap (\partial\Omega)_\delta,$$

where $\phi(x)$ is the point described by the couple $(y, -t)$ if x is described by (y, t). We define the varifold $\tilde{\mathcal{V}}$ by $\tilde{\mathcal{V}} := \mathcal{V}$ in Ω and $\tilde{\mathcal{V}} := \phi_\sharp \mathcal{V}$ in W, that is $\tilde{\mathcal{V}}$ consists in $\mathcal{V}$ union its reflection with respect to the boundary $\partial\Omega$. We then consider the manifold $\mathcal{M} := \Omega_\delta$ endowed with the smooth riemannian metric g defined by $g = g_0$ in $\bar{\Omega}$ and $g = \phi_*(g_0)$ in W, where g_0 is the euclidian metric on Ω.

Theorem 1. *Assume* (4) *and* (7)*. Let u_ε be a family of solutions of* (1)-(6) *satisfying the energy bound*

$$E_\varepsilon(u) \leq M|\log \varepsilon|$$

for a vector field $\vec{c}_\varepsilon$ satisfying

$$\vec{c}_\varepsilon \to 0 \qquad in\ \mathcal{C}^0(\bar{\Omega}) \quad as\ \varepsilon \to 0.$$

Then, the varifold $\mathcal{V}(\Sigma_{\mu_*}, \Theta_*)$ is stationary in Ω. Moreover, $\tilde{\mathcal{V}}$ is a stationary varifold in $(\mathcal{M}, g)$.

Remark 1. In the case where Ω is (locally) the half-plane $\mathbb{R}^N_+ = \mathbb{R}^*_+ \times \mathbb{R}^{N-1}$, then the theorem states that $\tilde{\mathcal{V}}$ is a stationary varifold in (locally) $\mathbb{R}^N$ for the usual metric.

This Theorem says that, in some weak sense, the union of the varifold $\mathcal{V}$ and its symmetric with respect to the boundary is "smooth", that is $\mathcal{V}$ must meet the boundary $\partial\Omega$ orthogonally. Since $\mathcal{V}$ is not in general a smooth curve, we may only use a weak formulation of this orthogonality. However, if $\mathcal{V}$ is a smooth curve up to the boundary, then Theorem 1 states that, denoting $\vec{\tau}$ the tangent unit vector to $\mathcal{V}$,

$$\vec{\tau} = \pm n \qquad \text{on } \partial\Omega.$$

The fact that the vortex must meet the boundary orthogonally can be found in the literature. For instance, in [CH], Chapman and Heron considered a domain which is the half-plane (in $\mathbb{R}^3$) $\{z < 0\}$ and a straight line vortex Γ, defined by $y = 0$, $x = mz \leq 0$ for a $0 \leq m < +\infty$, meeting the boundary $\{z = 0\}$ at 0. Using the London equation and the boundary conditions for the magnetic field, they proved, computing the propagation speed of the vortex at 0, that the coefficient m must be zero, for otherwise, the propagation speed would be infinite. However, their computation does not exclude the case of two vortices, defined by $y = 0$, $x = mz \leq 0$ and $y = 0$, $x = -mz \geq 0$, since in that case, the propagation is, due to the symmetry, zero. Our Theorem 1 states that there can not be another possibility involving two such coplanar straight lines vortices, that is vortices defined by $y = 0$, $x = mz \leq 0$ and $y = 0$, $x = -m'z \leq 0$ with $0 \leq m, m' < +\infty$ and $m \neq m'$ can not hold. Our Theorem even states that if we have two straight line vortices in the half plane $\{z < 0\}$ meeting at 0, then they must be in a plane orthogonal to $\{z = 0\}$. At the opposite of [CH], our approach is based on equation (1) only, whereas the London equation is the limit equation for the current (see (22) below), which is the second equation of the Ginzburg-Landau equation with magnetic field.

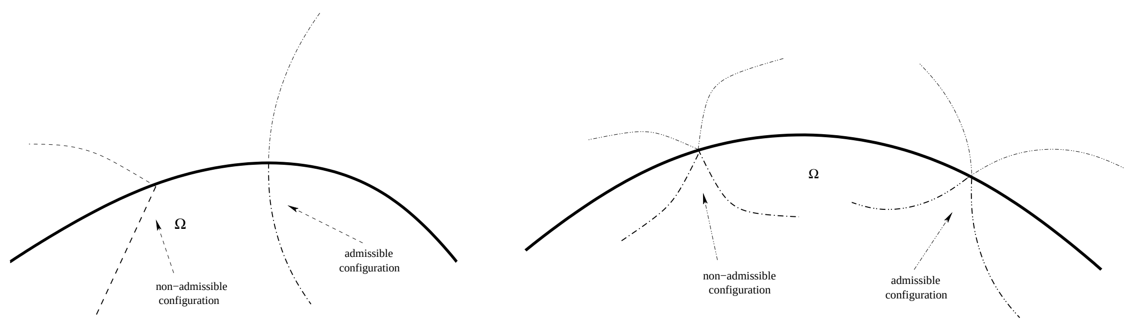
Remark 2. In the case $\vec{c}_\varepsilon \to \vec{c} \neq 0$ as $\varepsilon \to 0$, by Theorem 3 in [BOS], we know that the varifold $\mathcal{V}$ satisfies inside the domain Ω the curvature equation

$$H = \star\Big(\vec{c} \wedge (\star\frac{dJ_*}{d\mu_*})\Big), \tag{10}$$

where H is the generalized mean curvature of $\mathcal{V}$ and, up to a subsequence, J_* is a weak limit of the jacobian Ju_ε and $\star$ refers to Hodge duality. Theorem 1 generalizes then in the form (see Section 5)

$$\tilde{H} = \star\Big(\tilde{c} \wedge (\star\frac{d\tilde{J}_*}{d\tilde{\mu}_*})\Big), \tag{11}$$

where $\tilde{c}$, $\tilde{J}_*$ and $\tilde{\mu}_*$ are the extensions of $\vec{c}$, J_* and μ_* by reflection, and $\tilde{H}$ the generalized mean curvature of $\tilde{\mathcal{V}}$ in $(\mathcal{M}, g)$. Equation (11) also implies in somme weak sense that the vortex must be orthogonal to the boundary. We show in the figure below some non-admissible and admissible configurations for a vortex we assume "regular" (for instance $\frac{d\|J_*\|}{d\mu_*} = 1$), in the case where $\vec{c}$ can be non zero. In Theorem 1, the stationarity of $\mathcal{V}$ (thus of $\tilde{\mathcal{V}}$) inside the domain Ω is a direct consequence of (10) with $\vec{c} = 0$.



Non-admissible and admissible configurations (a) one vortex (b) two vortices.

1.2 Monotonicity and Clearing-Out Theorems

The second result is a Clearing-Out theorem for this equation. This result is also called η-compactness (in [R], [LR]) and η-ellipticity Lemma (in [BBO]). We recall the definition of the scaled energy, for a map $u : \Omega \to \mathbb{C}$,

$$\begin{aligned}\tilde{E}_\varepsilon(u, x_0, r) &:= \frac{1}{r^{N-2}} E_\varepsilon(u, \Omega \cap B_r(x_0)) = \frac{1}{r^{N-2}} \int_{\Omega\cap B_r(x_0)} e_\varepsilon(u) \\ &= \frac{1}{r^{N-2}} \int_{\Omega\cap B_r(x_0)} \frac{|\nabla u|^2}{2} + \frac{(a_\varepsilon(x) - |u|^2)^2}{4\varepsilon^2}\end{aligned}$$

and finally set

$$\check{B}_r(x_0) := \Omega \cap B_r(x_0)$$

and

$$r_\varepsilon := (\varepsilon^\mu |\log\varepsilon|)^{1/(N-1)},$$

where $\mu \in (0, 1)$ is a constant depending only on N.

We can now state our Clearing-Out result for the Dirichlet boundary condition (5). First, we make the following standard hypothesis for the boundary datum

$$|g_\varepsilon|_\infty \le C \quad \text{ and } \quad |\nabla g_\varepsilon|_\infty \le \frac{C}{\varepsilon}. \tag{12}$$

We also introduce the following quantity, for $0 \le r_1 \le r_2$ and $0 < \nu \le 1$,

$$T_\varepsilon^\nu(x_0, r_1, r_2) := \int_{r_1}^{r_2} \frac{1}{r^{N-2+\nu}} \int_{\partial\Omega\cap B_r(x_0)} e_\varepsilon^\top(g_\varepsilon)\, dr,$$

where

$$e_\varepsilon^\top(g_\varepsilon) := \frac{1}{2}\Big(|\nabla_\top g_\varepsilon|^2 + \frac{(a_\varepsilon(x) - |g_\varepsilon|^2)^2}{2\varepsilon^2}\Big).$$

Theorem 2. *Assume* (4) *and* (7)*. Let u be a solution of* (1)-(5) *on Ω, with g_ε satisfying* (12)*. Let $x_0 \in \bar\Omega$, $0 < \nu \le 1$, $r_\varepsilon^{1/2} \le r \le \min(R, (1+\Lambda_0)^{-1/\nu})$, where $R > 0$ depends only on Ω, and $\sigma > 0$ be given. Then, there exist constants $\eta > 0$ and $\varepsilon_0 > 0$ depending on σ, ν, N, Ω, Λ_0 and the constant C in* (12) *but independent of u and g_ε such that, for $\varepsilon \le \varepsilon_0$, if*

$$T_\varepsilon^\nu(x_0, \varepsilon, r_\varepsilon^{1/2}) \le \eta, \tag{13}$$

$$T_\varepsilon^\nu(x_0, r_\varepsilon, r) \le \eta|\log\varepsilon|, \tag{14}$$

and

$$\tilde{E}_\varepsilon(u, x_0, r) \le \eta|\log\varepsilon|, \tag{15}$$

then

$$|u(x_0)| \ge 1 - \sigma.$$

Note that one may take different ν's for (13) and (14), but we can always assume they are equal.

Remark 3. We emphasize that the quantity involved in T_ε^ν is related to the decay as $r \to 0$ of the scaled energy for g_ε, namely

$$\frac{1}{r^{N-3}} \int_{\partial\Omega\cap B_r(x_0)} e_\varepsilon^\top(g_\varepsilon).$$

We make an hypothesis at small scales ($r \le r_\varepsilon^{1/2}$) for (13), which is the suitable assumption for "g_ε is smooth enough and of modulus one", and an hypothesis at large scales (r can be of order one) for (14), which is an hypothesis on g_ε similar to the one made on u for (15).

Remark 4. If there exist $\delta \in (0, 2]$ and a constant $M > 0$ such that, for $0 < \rho \le r$,

$$\frac{1}{\rho^{N-3}} \int_{\partial\Omega\cap B_\rho(x_0)} |\nabla_\top g_\varepsilon|^2 + \frac{(a_\varepsilon(x) - |g_\varepsilon|^2)^2}{2\varepsilon^2} \le M\rho^\delta, \tag{16}$$

then, for $0 < r_1 \le r_2 \le r \le 1$ and $0 < \nu < \min(\delta, 1)$,

$$T_\varepsilon^\nu(x_0, r_1, r_2) \le M \int_{r_1}^{r_2} r^{\delta-\nu-1}\, dr \le M \frac{r_2^{\delta-\nu}}{\delta - \nu}.$$

Therefore, the hypothesis (13) is verified for ε sufficiently small (depending on ν, δ and M), and (14) is verified for $r \le 1$ and ε sufficiently small (depending on M). One may even consider for (16) a constant $M \ll |\log\varepsilon|$. In particular, if g_ε is uniformly M-lipschitzian of modulus 1 on $\partial\Omega \cap B_r(x_0)$, then by (8)

$$\frac{1}{\rho^{N-3}} \int_{\partial\Omega\cap B_\rho(x_0)} |\nabla_\top g_\varepsilon|^2 + \frac{(a_\varepsilon(x) - |g_\varepsilon|^2)^2}{2\varepsilon^2} \le CM^2\rho^2 + C\rho^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4 \le C(M^2 + \frac{1}{4})\rho^2,$$

where C depends only on Ω, thus (16) is satisfied with $\delta = 2$.

Remark 5. We would like to emphasize that we do not impose $|g_\varepsilon| \equiv 1$ (near the point x_0). The condition (13) (for $r \simeq \varepsilon$) however implies $|g_\varepsilon|(x_0) \simeq 1$ if x_0 is at distance less than ε from the boundary. We enlarge the conditions on the boundary datum already used in [LR] (and [BBO]). In this case, g_ε is a suitable smooth approximation of a map of modulus 1 smooth outside a finite union of smooth submanifolds of $\partial\Omega$ of dimension $N-3$. The Clearing-Out Theorem is then stated far away from these submanifolds.

Our Clearing-Out result for the the Neumann boundary condition (6) is the following.

Theorem 3. *Assume* (4) *and* (7). *Let u be a solution of* (1)-(6) *on Ω, $x_0 \in \bar\Omega$ and $\sigma > 0$ be given, and let $r_\varepsilon^{1/2} \le r \le \min(R, 1/(1+\Lambda_0))$, where $R > 0$ depends only on Ω. There exist constants $\eta > 0$ and $\varepsilon_0 > 0$, depending on N, Ω, σ and Λ_0 but independent of u, such that, for $0 < \varepsilon < \varepsilon_0$, if*

$$\tilde E_\varepsilon(u, x_0, r) \le \eta |\log \varepsilon|,$$

then

$$|u(x_0)| \ge 1 - \sigma.$$

Remark 6. These theorems do *not* give compactness on the solution u as $\varepsilon \to 0$. For the Dirichlet problem, the compactness properties follow from hypothesis on the *whole* boundary (see for instance [BBBO] for compactness in $W^{1,p}$, $1 \le p < N/(N-1)$).

These results rely strongly on monotonicity formulas of the scaled energy of solutions of (1). For the Dirichlet problem, the result is the following.

Proposition 1. *Assume* (4) *and* (7). *Let $\nu \in (0,1]$ and g_ε satisfying* (12). *There exist $R > 0$, depending only on Ω, $C > 0$, depending on Ω, ν and the constant C in* (12) *only, and $\beta > 0$ depending on N only such that, if u is a solution of* (1)-(5), *$0 < r \le \min(R, (1+\Lambda_0)^{-1/\nu})$ and $x_0 \in \bar\Omega$, then for any $0 < \theta < 1/2$, we have*

$$\tilde E_\varepsilon(x_0, \theta r) \le C\Big(\tilde E_\varepsilon(r) + T_\varepsilon^\nu(x_0, \theta r, r) + \Lambda_0 \varepsilon^\beta\Big). \tag{17}$$

For the Neumann problem, the result is the following.

Proposition 2. *Assume* (4) *and* (7). *There exist $\beta > 0$, $R > 0$ and $C > 0$ depending on Ω and N only such that, if u is a solution of* (1)-(6), *$x_0 \in \bar\Omega$ and $0 < r \le \min(R, 1/(1+\Lambda_0))$, then for any $0 < \theta < 1/2$,*

$$\tilde E_\varepsilon(x_0, \theta r) \le C\Big(\tilde E_\varepsilon(r) + \Lambda_0 \varepsilon^\beta\Big). \tag{18}$$

1.3 Models involving equation (1)

We would like to discuss some models involving equation (1), as well as the boundary conditions.

Note that (1) can be rewritten as

$$i|\log\varepsilon|\vec c(x)\cdot\nabla u = \Delta u + \frac{1}{\varepsilon^2} u(a_\varepsilon(x) - |u|^2), \tag{19}$$

where

$$a_\varepsilon(x) := 1 - d(x)\varepsilon^2|\log\varepsilon|^2.$$

When $\operatorname{div}\vec{c} = 0$, it is also equivalent to

$$(\nabla - i|\log\varepsilon|\frac{\vec{c}}{2})^2 u + \frac{1}{\varepsilon^2} u(b_\varepsilon(x) - |u|^2) = 0, \tag{20}$$

where

$$b_\varepsilon(x) := a_\varepsilon(x) + \varepsilon^2 |\log\varepsilon|^2 \frac{|\vec{c}(x)|^2}{4}.$$

If $\vec{c}|\log\varepsilon| = \vec{A}$ and $d = |\vec{c}|^2/4$ with $\operatorname{div}\vec{A} = 0$, then this equation is the first equation in the Ginzburg-Landau system of superconductivity, namely

$$(\nabla - i\vec{A}/2)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2). \tag{21}$$

The second equation for the induced magnetic field $H := |\log\varepsilon| \operatorname{curl}\vec{c}$ (in dimension $N = 3$) is

$$(iu, \nabla_A u) = \operatorname{curl} H + \operatorname{curl} H_{ex} = |\log\varepsilon| \operatorname{curl}^2 \vec{c} + \operatorname{curl} H_{ex}, \tag{22}$$

where H_{ex} is the imposed magnetic field and $\nabla_A = \nabla - i|\log\varepsilon|\vec{c}$ is the covariant derivative. Equations (21)-(22) are the Euler-Lagrange equations of the Ginzurg-Landau functional

$$J(u, \vec{c}) = \frac{1}{2} \int_\Omega \left|\nabla u - i|\log\varepsilon|\vec{c}u\right|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} + \left|H - H_{ex}\right|^2.$$

In this case, the natural boundary condition is

$$n \cdot (iu, \nabla_A u) = 0. \tag{23}$$

The functional J is gauge-invariant, that is, if $\psi \in H^2(\Omega)$, then

$$J(ue^{i\psi}, \vec{c} + \nabla\psi) = J(u, \vec{c}).$$

We can freeze the gauge-invariance by choosing, for instance, the Coulomb gauge

$$\begin{cases} \operatorname{div}\vec{c} & = 0 \quad \text{in } \Omega, \\ \vec{c} \cdot n & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In this case, the boundary condition (23) becomes with the Coulomb gauge

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

This justifies the study at the boundary with the homogeneous Neumann condition (6).

Writing (21) in the form (1) has the advantage to include in the same analysis the equation already mentioned (in dimension $N \geq 3$)

$$ic|\log\varepsilon|\partial_1 u = \Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2)$$

related to the travelling waves for the Gross-Pitaevskii equation with small speeds. This equation is used as a model for superfluidity, nonlinear optics and Bose-Einstein condensates. It is close to the Ginzburg-Landau equation (21) and a similar asymptotic analysis as $\varepsilon \to 0$ can be carried out for this equation.

We would like to mention that in [C], we have been interested in travelling vortex helices to the Gross-Pitaevskii equation. In this case, we approximate the problem on cylinders of axis x_1, and we impose the Dirichlet boundary condition $u = e^{i\theta}$ on the lateral surface of the cylinder that forces the solution u to have a degree one in the plane orthogonal to x_1. Therefore, this study required a Dirichlet boundary condition, whereas the Neumann condition is the natural one for the gauge-invariant functional J.

We refer to [BOS] for the generalization of the analysis of equation (1) inside the domain (see Theorems 2 and also 3 there). The proofs of Theorems 2 and 3 will follow the same lines as in appendix A of [BOS]. We also mention the study of minimizers in dimension 3 for the $U(1)$-Higgs model in [R]. For the study near the boundary for the Ginzburg-Landau functional without magnetic field ($d = |\vec{c}| \equiv 0$) and Dirichlet datum smooth outside a finite union of smooth $(N-2)$-dimensional submanifolds of $\partial\Omega$, we refer to [LR] (for minimizers in dimension $N \geq 3$) and [BBO] (for the general case).

The paper is organized as follows. In Section 2, we state and prove two lemmas concerning basic L^∞ bounds for u and ∇u. Section 3 is devoted to the monotonicity formulas and the proof of Propositions 1 and 2. In Section 4, we prove the Clearing-Out Theorems 2 and 3, while the result about the orthogonal anchoring of the vortex on the boundary of Theorem 1 is given in Section 5.

2 Basic L^∞ bounds

We first state two lemmas related to L^∞ bounds for u and ∇u. The first one concerns the Dirichlet problem.

Lemma 1. *Assume* (4) *and* (7). *Let u be a solution of* (1)-(5), *with g_ε satisfying* (12). *Then,*

$$|u|_\infty^2 \leq \max(|g_\varepsilon|_\infty^2, |b_\varepsilon|_\infty) \leq C \qquad \text{and} \qquad |\nabla u|_\infty \leq \frac{C}{\varepsilon}$$

for a constant C depending on Ω, Λ_0 and the constant C in (12) *only.*

The second one is for the Neumann problem.

Lemma 2. *Assume* (4) *and* (7). *Let $u \in H^1 \cap L^4(\Omega)$ be a solution of* (1)-(6). *Then, $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ for some $\alpha > 0$ and*

$$|u|_\infty^2 \leq |b_\varepsilon|_\infty \qquad \text{and} \qquad |\nabla u|_\infty \leq \frac{K}{\varepsilon},$$

where K depends on Ω and Λ_0.

In particular, for the Ginzburg-Landau functional with magnetic field (where $d \equiv |\vec{c}|^2/4$, thus $b_\varepsilon \equiv 1$), Lemma 2 states that $|u|_\infty \leq 1$.

2.1 Proof of Lemma 1

It is close to the proof of Lemma 3 in [BOS]. From (1), we deduce

$$\begin{aligned}\Delta|u|^2 &= 2(u,\Delta u)+2|\nabla u|^2 = -2\varepsilon^{-2}|u|^2(a_\varepsilon-|u|^2)+2|\log\varepsilon|(u,i\vec{c}\cdot\nabla u)+2|\nabla u|^2\\ &\geq -2\varepsilon^{-2}|u|^2(a_\varepsilon-|u|^2)-2|\vec{c}|\cdot|\log\varepsilon|\cdot|u|\cdot|\nabla u|+2|\nabla u|^2\\ &= -2\varepsilon^{-2}|u|^2(a_\varepsilon-|u|^2)+\Big(\sqrt{2}|\nabla u|-\frac{|\vec{c}|}{\sqrt{2}}|u|\cdot|\log\varepsilon|\Big)^2-\frac{|\vec{c}|^2}{2}|u|^2\cdot|\log\varepsilon|^2\\ &\geq -2\varepsilon^{-2}|u|^2\Big(|b_\varepsilon|_\infty-|u|^2\Big).\end{aligned}$$

Therefore, the function $w:=\max(|g_\varepsilon|_\infty^2,|b_\varepsilon|_\infty)-|u|^2$ satisfies

$$\begin{aligned}-\Delta w+2\varepsilon^{-2}|u|^2w\geq 0 &\quad \text{in } \Omega,\\ w\geq 0 &\quad \text{on } \partial\Omega,\end{aligned}$$

and by the maximum principle, we deduce

$$w\geq 0 \quad \text{in } \Omega.$$

Concerning the bound on the gradient, we consider the scaled map $\hat{u}(x):=u(\varepsilon x)$, which satisfies

$$\begin{aligned}\Delta\hat{u}+\hat{u}(\hat{a}_\varepsilon-|\hat{u}|^2)=i\varepsilon|\log\varepsilon|\hat{c}\cdot\nabla\hat{u} &\qquad \text{in } \frac{\Omega}{\varepsilon},\\ \hat{u}=g_\varepsilon(\varepsilon x) &\qquad \text{on } \frac{\partial\Omega}{\varepsilon},\end{aligned}$$

where $\hat{c}(x):=c(\varepsilon x)$ and $\hat{d}(x):=d(\varepsilon x)$. By standard elliptic estimates (see [GT]), since $|\nabla(g_\varepsilon(\varepsilon x))|\leq C$ by hypothesis (12),

$$|\nabla\hat{u}|_{L^\infty}\leq C$$

for a constant C depending on Ω and Λ_0, and the estimate for u is obtained by scaling back. □

2.2 Proof of Lemma 2

The proof of the $\mathcal{C}^{2,\alpha}$ regularity of u uses a standard bootstrap argument and the fact that the coefficients c,d are lipschitzian. Concerning the L^∞ bounds, as in the proof of Lemma 1, we find that $w:=|b_\varepsilon|_\infty-|u|^2$ satisfies

$$\begin{aligned}-\Delta w+2\varepsilon^{-2}|u|^2w\geq 0 &\quad \text{in } \Omega,\\ \frac{\partial w}{\partial n}=0 &\quad \text{on } \partial\Omega,\end{aligned}$$

since $\frac{\partial w}{\partial n}=-2(u,\frac{\partial u}{\partial n})=0$ by (6). We then adapt an idea of [F], used in the proof of universal bounds for travelling waves for the Gross-Pitaevskii equation. For $f:\Omega\to\mathbb{R}$, we decompose $f=f^+-f^-$ in its nonpositive and nonnegative part ($f^+,f^-\geq 0$, $f^+f^-=0$). Since w and Δw are Hölder continuous, we have by Kato's inequality (see [B], [K])

$$\Delta(w^-)=\Delta((-w)^+)\geq \mathrm{sign}^+(-w)\Delta(-w)\geq 2\frac{|u|^2}{\varepsilon^2}\mathrm{sign}^+(-w)(-w). \tag{24}$$

Therefore (if $w \geq 0$, the right-hand side of (24) is zero, and if $w < 0$, then $|u|^2 > |b_\varepsilon|_\infty \geq 0$),

$$\Delta(w^-) \geq \frac{2|b_\varepsilon|_\infty}{\varepsilon^2} w^- \geq 0. \tag{25}$$

From (25), it is clear that we can not have $w^- \equiv cte > 0$, since $|b_\varepsilon|_\infty > 0$. As a consequence, in view of (25), we deduce by the strong maximum principle (Ω is connected) that either $w^- = cte$, and then this constant must be zero, either w^- achieves its maximum only *on* the boundary, for instance at $x_0 \in \partial\Omega$. Assuming $w^- \not\equiv 0$, we have $w^-(x_0) > 0$. In particular, since $w \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$, in a neighborhood of x_0 in $\bar{\Omega}$, $w^- = -w > 0$ is $\mathcal{C}^{2,\alpha}$. It is then well-known that in this case, since $w^- > 0$ in this neighborhood, we have by (25),

$$-\frac{\partial w}{\partial n} = \frac{\partial w^-}{\partial n} > 0.$$

This contradicts the boundary condition $\frac{\partial w}{\partial n} = 0$. Therefore, $w^- \equiv 0$ and $w = w^+ \geq 0$, that is $|u|^2_\infty \leq |b_\varepsilon|_\infty$, which finishes the proof for the L^∞ bound. For the estimate on the gradient, we consider the scaled map $\hat{u}(x) := u(\varepsilon x)$, which satisfies

$$\begin{aligned} \Delta\hat{u} + \hat{u}(\hat{a}_\varepsilon - |\hat{u}|^2) &= i\varepsilon|\log\varepsilon|\hat{c}\cdot\nabla\hat{u} && \text{in } \frac{\Omega}{\varepsilon}, \\ \frac{\partial\hat{u}}{\partial n} &= 0 && \text{on } \frac{\partial\Omega}{\varepsilon}. \end{aligned}$$

By standard elliptic estimates, we have

$$|\nabla\hat{u}|_{L^\infty} \leq C$$

and we conclude by scaling back. □

3 Monotonicity formulas at the boundary

As already mentioned, we follow the lines of the proof of Theorem 2 of [BOS] given in appendix A there. When this will not lead to a confusion, we will denote $\tilde{E}_\varepsilon(u, x_0, r)$ and $\breve{B}_r(x_0)$ by $\tilde{E}_\varepsilon(x_0, r)$, or even $\tilde{E}_\varepsilon(r)$, and $\breve{B}_r$. We first recall the Pohozaev identity.

Lemma 3.1. *Let u be a solution of* (1) *on Ω, then for any $z_0 \in \mathbb{R}^N$ and $\omega \subset \Omega$,*

$$\frac{N-2}{2}\int_\omega |\nabla u|^2 + \frac{N}{4\varepsilon^2}\int_\omega (a_\varepsilon(x) - |u|^2)^2 - \frac{N-1}{2}|\log\varepsilon|\int_\omega \langle Ju, \sum_i c_i(x)\xi_i(x - z_0)\rangle$$

$$= \int_{\partial\omega}\Big[(x - z_0)\cdot n e_\varepsilon(u) - (\frac{\partial u}{\partial n}, (x - z_0)\cdot\nabla u)\Big] + \frac{|\log\varepsilon|^2}{2}\int_\omega (a_\varepsilon(x) - |u|^2)(x - z_0)\cdot\nabla d(x).$$

Here, ξ_i stands for the 2-form

$$\xi_i := \frac{2}{N-1}\sum_{j\neq i} x_j dx_i \wedge dx_j.$$

3.1 The Dirichlet problem

In this subsection, we assume that u is a solution to the Dirichlet problem (1)-(5). We will denote, for $r>0$,

$$G_\varepsilon(x_0,r) := \frac{1}{2r^{N-2}}\int_{\partial\Omega\cap B_r(x_0)} |\nabla_\top g_\varepsilon|^2 + \frac{(a_\varepsilon(x)-|g_\varepsilon|^2)^2}{2\varepsilon^2} = \frac{1}{r^{N-2}}\int_{\partial\Omega\cap B_r(x_0)} e_\varepsilon^\top(g_\varepsilon).$$

Note that G_ε is *not* the scaled energy for g_ε. We fix $0<\nu\le 1$. In the sequel, C denotes a constant depending on N, Ω and ν only.

Lemma 3.2. *Let u be a solution of (1)-(5) on Ω, then for $r>0$ and $x_0\in\bar\Omega$*

$$\begin{aligned}\frac{d}{dr}(\tilde E_\varepsilon(x_0,r)) =& \frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r(x_0)}\Big|\frac{\partial u}{\partial n}\Big|^2 + \frac{1}{r^{N-1}}\int_{B_r(x_0)\cap\partial\Omega}\frac12(x-x_0)\cdot n\Big|\frac{\partial u}{\partial n}\Big|^2\\ &+\frac{1}{r^{N-1}}\int_{\check B_r(x_0)}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2}\\ &-\frac{N-1}{2r^{N-1}}|\log\varepsilon|\int_{\check B_r(x_0)}\langle Ju,\sum_i c_i(x)\xi_i(x-x_0)\rangle \\ &-\frac{|\log\varepsilon|^2}{2r^{N-1}}\int_{\check B_r(x_0)}(a_\varepsilon(x)-|u|^2)(x-x_0)\cdot\nabla d(x)\\ &-\frac{1}{r^{N-1}}\int_{B_r(x_0)\cap\partial\Omega}(x-x_0)\cdot n e_\varepsilon^\top(g_\varepsilon) - (\frac{\partial u}{\partial n},(x-x_0)_\top\cdot\nabla_\top g_\varepsilon),\end{aligned} \tag{26}$$

where $(x-x_0)_\top$ is the orthogonal projection of $x-x_0$ on the tangent hyperplane to $\partial\Omega$ at x.

Proof. Up to a translation, we may assume $x_0=0$. One has

$$\begin{aligned}\frac{d\tilde E_\varepsilon}{dr} &= -\frac{N-2}{r^{N-1}}E_\varepsilon(r) + \frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r}\frac{|\nabla u|^2}{2}+\frac{(a_\varepsilon(x)-|u|^2)^2}{4\varepsilon^2}\\ &= -\frac{1}{r^{N-1}}\Big(\frac{N-2}{2}\int_{\check B_r}|\nabla u|^2 + \frac{N}{4\varepsilon^2}\int_{\check B_r}(a_\varepsilon(x)-|u|^2)^2\Big)\\ &\quad+\frac{1}{r^{N-1}}\int_{\check B_r}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2}+\frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r}e_\varepsilon(u).\end{aligned}$$

We use Lemma 3.1 (with $\omega=\check B_r(x_0)$ and $z_0=x_0$) for the first term and then split $\partial\check B_r$ into

$$\partial\check B_r = (B_r\cap\partial\Omega)\mathring{\cup}(\Omega\cap\partial B_r)$$

to obtain, since $x \cdot n = r$ on $\Omega \cap \partial B_r$,

$$\begin{aligned}
\frac{d\tilde{E}_\varepsilon}{dr} &= \frac{1}{r^{N-1}} \int_{\check{B}_r} \frac{(a_\varepsilon(x) - |u|^2)^2}{2\varepsilon^2} + \frac{1}{r^{N-2}} \int_{\Omega \cap \partial B_r} e_\varepsilon(u) \\
&\quad - \frac{N-1}{2r^{N-1}} |\log \varepsilon| \int_{\check{B}_r} \langle Ju, \sum_i c_i(x) \xi_i(x) \rangle \\
&\quad - \frac{|\log \varepsilon|^2}{2r^{N-1}} \int_{\check{B}_r} (a_\varepsilon(x) - |u|^2) x \cdot \nabla d(x) \\
&\quad - \frac{1}{r^{N-1}} \int_{\partial \check{B}_r} x \cdot n e_\varepsilon(u) - (\frac{\partial u}{\partial n}, x \cdot \nabla u), \\
&= \frac{1}{r^{N-1}} \int_{\check{B}_r} \frac{(a_\varepsilon(x) - |u|^2)^2}{2\varepsilon^2} + \frac{1}{r^{N-2}} \int_{\Omega \cap \partial B_r} \Big| \frac{\partial u}{\partial n} \Big|^2 \\
&\quad - \frac{N-1}{2r^{N-1}} |\log \varepsilon| \int_{\check{B}_r} \langle Ju, \sum_i c_i(x) \xi_i(x) \rangle \\
&\quad - \frac{|\log \varepsilon|^2}{2r^{N-1}} \int_{\check{B}_r} (a_\varepsilon(x) - |u|^2) x \cdot \nabla d(x) \qquad (27) \\
&\quad - \frac{1}{r^{N-1}} \int_{B_r \cap \partial \Omega} x \cdot n e_\varepsilon(u) - (\frac{\partial u}{\partial n}, x \cdot \nabla u).
\end{aligned}$$

It suffices then to write, on $B_r \cap \partial \Omega$,

$$e_\varepsilon(u) = \frac{1}{2} \Big| \frac{\partial u}{\partial n} \Big|^2 + \frac{1}{2} |\nabla_\top g_\varepsilon|^2 + \frac{(a_\varepsilon(x) - |g_\varepsilon(x)|^2)^2}{4\varepsilon^2} = \frac{1}{2} \Big| \frac{\partial u}{\partial n} \Big|^2 + e_\varepsilon^\top(g_\varepsilon)$$

and

$$x = (x \cdot n)n + x_\top,$$

thus

$$(\frac{\partial u}{\partial n}, x \cdot \nabla u) = x \cdot n \Big| \frac{\partial u}{\partial n} \Big|^2 + (\frac{\partial u}{\partial n}, x_\top \cdot \nabla_\top g_\varepsilon),$$

to finally deduce

$$x \cdot n e_\varepsilon(u) - (\frac{\partial u}{\partial n}, x \cdot \nabla u) = x \cdot n e_\varepsilon^\top(g_\varepsilon) - \frac{x \cdot n}{2} \Big| \frac{\partial u}{\partial n} \Big|^2 - (\frac{\partial u}{\partial n}, x_\top \cdot \nabla_\top g_\varepsilon).$$

Inserting this in the last integral yields (26). $\square$

We note in equality (26) the last term involving the normal derivative of u. The next lemma provides an estimate for this term.

Lemma 3.3. (Control of the normal derivative). *Let u be a solution of* (1)-(5). *There exist C and R depending only on Ω such that, for all $x_0 \in \bar{\Omega}$ and $0 < r < R$, there exists $z_0 \in \check{B}_r(x_0)$ such that*

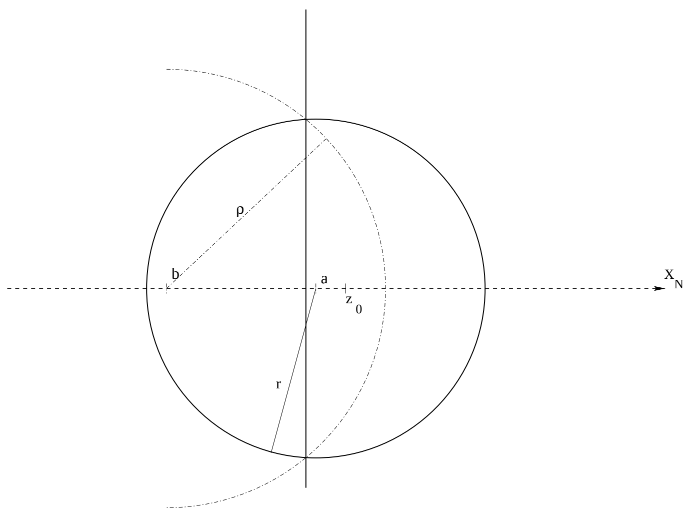
$$\begin{aligned}
r \int_{\partial \Omega \cap B_r(x_0)} \Big| \frac{\partial u}{\partial n} \Big|^2 \leq C \Big(\int_{\check{B}_r(x_0)} e_\varepsilon(u) + r \int_{\partial \Omega \cap B_r(x_0)} e_\varepsilon^\top(g_\varepsilon) + |\log \varepsilon| \cdot \Big| \int_\omega \langle Ju, \sum_i c_i(x) \xi_i(x - z_0) \rangle \Big| \\
+ \frac{|\log \varepsilon|^2}{2} \Big| \int_\omega (a_\varepsilon - |u|^2)(x - z_0) \cdot \nabla d \Big| \Big), \qquad (28)
\end{aligned}$$

where $\omega \subset \check{B}_r(x_0)$ depends on u, x_0 and r.

The proof is, as in [LR], based on a Pohozaev identity at a point z_0 around which $\check{B}_r(x_0)$ is strictly starshaped. However, we will not use a "good" extension of g inside the domain Ω as in [LR] (see Lemma II.5 there), since it requires a strong regularity hypothesis (for instance, g_ε bounded in $\mathcal{C}^{1,1}$ around x_0) and will not enable us to treat the case of the monotonicity at large scale (see Remark 3.1 below).

Proof. For simplicity, assume first that $\partial\Omega$ is locally the half-plane $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times\{0\}$. We also assume (up to a translation) $x_0 = (0,\dots,0,a)$. We assume first that $0 \le a \le r/4$, that is x_0 is close to the boundary $\partial\Omega$. We define $y := (0,\dots,0,b)$ for $b \le a$ and $\rho := (r^2 - a^2 + b^2)^{1/2}$. The intersection of $\partial\mathbb{R}^N_+$ and the balls $B_\rho(y)$ and $B_r(x_0)$ is the ball in $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times\{0\}$ centered at 0 and of radius $(r^2-a^2)^{1/2}$. By averaging, there exists $r' \in (r, 9r/8)$ such that, for $\rho = r'$ and $b = -(r'^2 - r^2 + a^2)^{1/2}$,

$$\int_{\mathbb{R}^N_+\cap\partial B_{r'}(y)} e_\varepsilon(u) \le \frac{C}{r}\int_{\check{B}_r(x_0)} e_\varepsilon(u). \tag{29}$$



Moreover, since $r' \le 9r/8$ and $0 \le a < r/4$, $b^2 \le 21r^2/64$, then $r' + b \ge r - \sqrt{21}r/8 \ge r/3$. We then set $z_0 := (0,\dots,0,\frac{r'+b}{2}) \in \check{B}_r(x_0)$ and easily see that, since $a \le r/4$, $\omega := \check{B}_r(x_0)\cap B_{r'}(y)$ is strictly starshaped around z_0, that is there exists $\alpha > 0$ such that

$$(x - z_0)\cdot n \ge \alpha r. \tag{30}$$

Next, we apply the Pohozaev identity of Lemma 3.1 with z_0, x_0 and ω to obtain

$$\frac{N-2}{2}\int_\omega |\nabla u|^2 + \frac{N}{4\varepsilon^2}\int_\omega (a_\varepsilon(x) - |u|^2)^2 - \frac{N-1}{2}|\log\varepsilon|\int_\omega \langle Ju, \sum_i c_i(x)\xi_i(x-z_0)\rangle$$

$$= \int_{\partial\omega}\Big[(x-z_0)\cdot n e_\varepsilon(u) - (\frac{\partial u}{\partial n}, (x-z_0)\cdot\nabla u)\Big] + \frac{|\log\varepsilon|^2}{2}\int_\omega (a_\varepsilon(x) - |u|^2)(x-z_0)\cdot\nabla d(x).$$

As in the proof of Lemma 3.2, we write $x - z_0 = ((x-z_0)\cdot n)n + (x-z_0)_\top$ to deduce

$$C\int_\omega e_\varepsilon(u) + C|\log\varepsilon|\cdot\Big|\int_\omega \langle Ju, \sum_i c_i(x)\xi_i(x-z_0)\rangle\Big| + |\log\varepsilon|^2\Big|\int_\omega (a_\varepsilon(x)-|u|^2)(x-z_0)\cdot\nabla d(x)\Big|$$

$$\geq \int_{\partial\omega}(x-z_0)\cdot n\Big|\frac{\partial u}{\partial n}\Big|^2 - (x-z_0)\cdot n|\nabla_\top u|^2 + 2(\frac{\partial u}{\partial n}, (x-z_0)_\top\cdot\nabla_\top u) - (x-z_0)\cdot n\frac{(a_\varepsilon-|u|^2)^2}{2\varepsilon^2}.$$

In the last integral, in view of the starshapedness assumption (30), the third term has an absolute value

$$\leq \frac{1}{2}(x-z_0)\cdot n\Big|\frac{\partial u}{\partial n}\Big|^2 + Cr|\nabla_\top u|^2.$$

Thus, using the starshapedness assumption (30) and splitting $\partial\omega$ into $\partial\Omega\cap B_r(x_0)$ and $\Omega\cap\partial B_{r'}(y)$,

$$\begin{aligned}C\int_\omega e_\varepsilon(u) + C|\log\varepsilon|\cdot\Big|\int_\omega \langle Ju, \sum_i c_i(x)\xi_i(x-z_0)\rangle\Big| + |\log\varepsilon|^2\Big|\int_\omega (a_\varepsilon(x)-|u|^2)(x-z_0)\cdot\nabla d(x)\Big| \\ \geq \frac{1}{2}\int_{\partial\omega}(x-z_0)\cdot n\Big|\frac{\partial u}{\partial n}\Big|^2 - Cr(|\nabla_\top u|^2 + \frac{(a_\varepsilon-|u|^2)^2}{2\varepsilon^2}) \\ \geq \frac{\alpha r}{2}\int_{\partial\omega}\Big|\frac{\partial u}{\partial n}\Big|^2 - Cr\int_{\partial\Omega\cap B_r(x_0)} e_\varepsilon^\top(g_\varepsilon) - Cr\int_{\Omega\cap\partial B_{r'}(y)} e_\varepsilon(u).\end{aligned}$$

We conclude estimating the last term by (29). We assume now that $a \geq r/4$, that is x_0 is far enough from the boundary. Then, we have $n = -\vec{e}_N$ and, if $x \in \partial\mathbb{R}^N_+$, then $x_N = 0$ and

$$(x-x_0)\cdot n = a - x_N = a \geq \frac{r}{4}. \tag{31}$$

In other words, $\breve{B}_r(x_0)$ is strictly starshaped around x_0. We conclude then as in the previous case. This concludes the proof in the case Ω is locally an half-plane. For the general case, we use local charts and note that the starshapedness assumptions (30) or (31) will still be true at least for $r < R$ (depending on Ω). $\square$

We then prove a first monotonicity formula useful for small scales, which is the boundary version of Lemma 4 in [BOS]. Note that the presence of the term r^ν in front of Λ and $r^{-\nu}$ in front of $G_\varepsilon(r)$ is specific to the Dirichlet condition.

Lemma 3.4. (Monotonicity at small scales). *There exist C and $0 < R < 1$, depending only on ν and Ω, such that for any solution u of (1)-(5) on Ω, denoting*

$$\Lambda := C(1+\Lambda_0|\log\varepsilon|) \quad \textit{and} \quad Q := C\Lambda_0\varepsilon|\log\varepsilon|^2,$$

and any $x_0 \in \bar\Omega$ and $0 < r \leq \min(R, \Lambda^{-1/\nu}) \leq 1$, we have

$$\begin{aligned}\frac{d}{dr}\Big(\exp(\Lambda r^\nu)(\tilde{E}_\varepsilon(r) + \frac{Q^2}{\Lambda})\Big) \geq \frac{1}{r^{N-1}}\int_{\Omega\cap\partial B_r(x_0)}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} + \frac{1}{r^{N-1}}\int_{B_r(x_0)\cap\partial\Omega}\frac{|x\cdot n|}{2}\Big|\frac{\partial u}{\partial n}\Big|^2 \\ + \frac{1}{r^{N-2}}\int_{\partial B_r(x_0)\cap\Omega}\Big|\frac{\partial u}{\partial n}\Big|^2 - C\frac{G_\varepsilon(r)}{r^\nu}.\end{aligned}$$

Proof. First, we note that, if $0 < r \leq R(\Omega)$ is sufficiently small, then for all $x \in \breve{B}_r(x_0)$,

$$(x-x_0)\cdot n \geq |(x-x_0)\cdot n| - Cr^2, \tag{32}$$

where C depends on Ω only. This fact was already used in [LR] (Lemma II.5). We recall the argument. One may assume that, for $r < R$ sufficiently small, $\check{B}_r$ is the uppergraph of $\psi : B_1(0) \subset \mathbb{R}^{N-1} \to \mathbb{R}$ and that $\psi(0) = |\nabla\psi(0)| = 0$, so that the tangent hyperplane at Ω at $\psi(0) = 0$ is $\mathbb{R}^{N-1} \times \{0\}$. Therefore, the outward normal writes

$$n = (1 + |\nabla\psi|^2)^{-1/2}\Big(\vec{e}_N - \sum_{i=1}^{N-1} \partial_i\psi\vec{e}_i\Big),$$

where $(\vec{e}_i)_{1\leq i\leq N}$ is the canonical basis of $\mathbb{R}^N$. In order to prove (32), it suffices then to prove

$$(x - x_0) \cdot \vec{e}_N \geq -Cr^2,$$

since $\nabla\psi(0) = 0$, so $|\nabla\psi| \leq Cr$. This last inequality is a direct consequence of the fact that $T_0(\partial\Omega) = \mathbb{R}^{N-1} \times \{0\}$ and Ω is locally the *uppergraph* of ψ. We now turn to the proof of Lemma 3.4.

Once more, we assume $x_0 = 0$. We have to estimate each term on the right hand side of (26). For the fourth one, we use the rough estimate for the jacobian

$$\|Ju(x)\| \leq C|\nabla u(x)|^2 \quad \text{and} \quad \|\xi_i(x)\| \leq Cr \quad \text{for all } x \in B_r,$$

which yields

$$\begin{aligned}\Big|\frac{N-1}{2r^{N-1}}|\log\varepsilon| \int_{\check{B}_r(x_0)} \langle Ju, \sum_i c_i(x)\xi_i(x)\rangle\Big| &\leq \frac{C}{r^{N-2}}|\vec{c}|_\infty|\log\varepsilon| \int_{\check{B}_r} |\nabla u(x)|^2 \\ &\leq C\Lambda_0|\log\varepsilon|\tilde{E}_\varepsilon(r). \end{aligned} \tag{33}$$

For the fifth one, by Cauchy-Schwarz,

$$\begin{aligned}\Big|\frac{|\log\varepsilon|^2}{2r^{N-1}} \int_{\check{B}_r} (a_\varepsilon(x) - |u|^2)x \cdot \nabla d(x)\Big| &\leq \frac{C}{r^{N-2}}\Lambda_0\varepsilon|\log\varepsilon|^2 \int_{\check{B}_r} \frac{|a_\varepsilon(x) - |u|^2|}{\varepsilon} \\ &\leq \frac{C}{r^{N-2}}\Lambda_0\varepsilon|\log\varepsilon|^2 r^{N/2}\Big(\int_{\check{B}_r} \frac{(a_\varepsilon(x) - |u|^2)^2}{\varepsilon^2}\Big)^{1/2} \\ &\leq Cr\Lambda_0\varepsilon|\log\varepsilon|^2\tilde{E}_\varepsilon(r)^{1/2} \\ &\leq \tilde{E}_\varepsilon(r) + Cr^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4. \end{aligned} \tag{34}$$

Concerning the second term, we have by (32)

$$\frac{1}{r^{N-1}} \int_{B_r\cap\partial\Omega} \frac{1}{2}x \cdot n\Big|\frac{\partial u}{\partial n}\Big|^2 \geq \frac{1}{2r^{N-1}} \int_{B_r\cap\partial\Omega} |x \cdot n| \cdot \Big|\frac{\partial u}{\partial n}\Big|^2 - \frac{C}{r^{N-3}} \int_{B_r\cap\partial\Omega} \Big|\frac{\partial u}{\partial n}\Big|^2. \tag{35}$$

We use (28) of Lemma 3.3 to estimate

$$\begin{aligned} r\int_{\partial\Omega\cap B_r} \Big|\frac{\partial u}{\partial n}\Big|^2 \leq C\Big(&\int_{\check{B}_r} e_\varepsilon(u) + |\log\varepsilon| \cdot \Big|\int_\omega \langle Ju, \sum_i c_i(x)\xi_i(x - z_0)\rangle\Big| \\ &+ r\int_{\partial\Omega\cap B_r} e_\varepsilon^\top(g_\varepsilon) + \frac{|\log\varepsilon|^2}{2}\Big|\int_\omega (a_\varepsilon - |u|^2)(x - z_0) \cdot \nabla d\Big|\Big). \end{aligned} \tag{36}$$

In (36), we estimate the second term as in (33) and the fourth one as in (34) (since $\omega \subset \check{B}_r$) to obtain

$$\frac{1}{r^{N-3}} \int_{\partial\Omega\cap B_r} \Big|\frac{\partial u}{\partial n}\Big|^2 \leq C\Big(\tilde{E}_\varepsilon(r) + rG_\varepsilon(r) + \Lambda_0|\log\varepsilon|\tilde{E}_\varepsilon(r) + Cr^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4\Big). \tag{37}$$

Inserting (37) in (35) yields

$$\frac{1}{2r^{N-1}}\int_{B_r\cap\partial\Omega} x\cdot n\Big|\frac{\partial u}{\partial n}\Big|^2 \geq \frac{1}{2r^{N-1}}\int_{B_r\cap\partial\Omega}|x\cdot n|\cdot\Big|\frac{\partial u}{\partial n}\Big|^2 - \Lambda\tilde{E}_\varepsilon(r) - CrG_\varepsilon(r) - Cr^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4, \tag{38}$$

where $\Lambda = C(1+\Lambda_0|\log\varepsilon|)$. For the last term in (26), we have first

$$\frac{1}{r^{N-1}}\int_{B_r\cap\partial\Omega} x\cdot n e_\varepsilon^\top(g_\varepsilon) \leq G_\varepsilon(r)$$

and since $|x|\leq r$,

$$\begin{aligned}\Big|\frac{1}{r^{N-1}}\int_{B_r\cap\partial\Omega}(\frac{\partial u}{\partial n}, x_\top\cdot\nabla_\top g_\varepsilon)\Big| &\leq \frac{1}{r^{N-2}}\int_{B_r\cap\partial\Omega}\Big|\frac{\partial u}{\partial n}\Big|\cdot|\nabla_\top g_\varepsilon| \\ &\leq \frac{r^{1-\nu}}{\nu r^{N-1}}\int_{B_r\cap\partial\Omega} e_\varepsilon^\top(g_\varepsilon) + \nu\frac{r^{\nu-1}}{r^{N-3}}\int_{B_r\cap\partial\Omega}\frac{1}{2}\Big|\frac{\partial u}{\partial n}\Big|^2\end{aligned}$$

which yields, using (37), the estimate of the last term in (26)

$$\frac{1}{r^{N-1}}\Big|\int_{\partial\Omega\cap B_r} x\cdot n e_\varepsilon^\top(g_\varepsilon) - (\frac{\partial u}{\partial n}, x_\top\cdot\nabla_\top g_\varepsilon)\Big| \leq \frac{\nu\Lambda}{r^{1-\nu}}\tilde{E}_\varepsilon(r) + C\frac{G_\varepsilon(r)}{r^\nu} + Cr^{2-\nu}\Lambda_0^2\varepsilon^2|\log\varepsilon|^4. \tag{39}$$

Inserting estimates (33), (34), (35), (38) and (39) into (26) gives

$$\begin{aligned}\frac{d\tilde{E}_\varepsilon}{dr} \geq &\frac{1}{2r^{N-1}}\int_{B_r\cap\partial\Omega}|x\cdot n|\cdot\Big|\frac{\partial u}{\partial n}\Big|^2 + \frac{1}{r^{N-1}}\int_{\breve{B}_r}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} \\ &+ \frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r}\Big|\frac{\partial u}{\partial n}\Big|^2 - \frac{\nu\Lambda}{r^{1-\nu}}\tilde{E}_\varepsilon(r) - C\frac{G_\varepsilon(r)}{r^\nu} - Cr^{2-\nu}\Lambda_0^2\varepsilon^2|\log\varepsilon|^4,\end{aligned}$$

from which we infer for $r\leq\Lambda^{-1/\nu}$ (note that $r^{2-\nu}\leq r^{\nu-1}$ since $0<\nu\leq 1$)

$$\begin{aligned}\frac{d}{dr}\Big(\exp(\Lambda r^\nu)\tilde{E}_\varepsilon\Big) &= \exp(\Lambda r^\nu)\frac{d\tilde{E}_\varepsilon}{dr} + \frac{\nu\Lambda}{r^{1-\nu}}\exp(\Lambda r^\nu)\tilde{E}_\varepsilon(r) \\ &\geq \frac{1}{2r^{N-1}}\int_{B_r\cap\partial\Omega}|x\cdot n|\cdot\Big|\frac{\partial u}{\partial n}\Big|^2 + \frac{1}{r^{N-1}}\int_{\breve{B}_r}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} \\ &\quad + \frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r}\Big|\frac{\partial u}{\partial n}\Big|^2 - C\exp(\Lambda r^\nu)\frac{G_\varepsilon(r)}{r^\nu} - C\exp(\Lambda r^\nu)r^{\nu-1}\Lambda_0^2\varepsilon^2|\log\varepsilon|^4 \\ &\geq \frac{1}{2r^{N-1}}\int_{B_r\cap\partial\Omega}|x\cdot n|\cdot\Big|\frac{\partial u}{\partial n}\Big|^2 + \frac{1}{r^{N-1}}\int_{\breve{B}_r}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} \\ &\quad + \frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r}\Big|\frac{\partial u}{\partial n}\Big|^2 - C\frac{G_\varepsilon(r)}{r^\nu} - \frac{d}{dr}\Big(\frac{Q^2}{\Lambda}\exp(\Lambda r^\nu)\Big)\end{aligned}$$

and the proof is complete. □

The previous monotonicity formula is useful for $r\leq C(1+\Lambda_0|\log\varepsilon|)^{-1/\nu} = \Lambda^{-1/\nu}$, that is r small if $\Lambda_0>0$. As in [BOS], the monotonicity formula for large scales will be a consequence of the refined estimates on jacobians as in [JS].

Lemma 3.5. (Jerrard & Soner). *Assume $u \in H^1_{loc}(\Omega, \mathbb{C})$, $\varphi \in \mathcal{C}^{0,1}_c(\Omega, \Lambda^2\mathbb{R}^N)$. There exist K and $\alpha \in (0,1)$, depending only on N and $|\Omega|$, such that, denoting $\mathcal{K} := Supp(\varphi)$,*

$$\Big|\int_\Omega \langle Ju, \varphi\rangle\Big| \leq \frac{K}{|\log\varepsilon|}|\varphi|_\infty E_\varepsilon(u,\mathcal{K}) + K\varepsilon^\alpha|d\varphi|_\infty(1+|\mathcal{K}|^2)(1+E_\varepsilon(u,\mathcal{K})). \tag{40}$$

The advantage of this estimate is the factor $|\log\varepsilon|$ dividing the energy. Note that this lemma is stated with the energy E_ε and not the usual Ginzburg-Landau energy used in [JS] (corresponding to $d \equiv 0$), but these two energies are close with our hypothesis, since one may infer from $0 \leq d \leq \Lambda_0$ that

$$\Big|\int_\Omega \frac{(1-|u|^2)^2}{2\varepsilon^2} - \int_\Omega \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2}\Big| \leq |\Omega|\Lambda_0^2\varepsilon|\log\varepsilon|^4 + \varepsilon\int_\Omega \frac{(1-|u|^2)^2}{2\varepsilon^2}.$$

Remark 3.1. We emphasize that this is the Pohozaev identity we used for Lemma 3.3 which provides the control of the normal derivative using the estimate of Jerrard and Soner of Lemma 3.5. The extension procedure of [LR] would have led to a term

$$|\log\varepsilon|\int_{\check{B}_r}\sum_{k,l=1}^N (ic_k(x)\partial_k u, x_l\partial_l\overline{g}(x)),$$

where $\overline{g}$ is a "good" extension inside Ω of g, and this term would be difficult to handle since it is not a jacobian if $\overline{g} \neq u$, thus we do not expect a compensation property.

For our purpose, we will need for our study a boundary version of this result, in order to have an estimate close to (40) for a φ having a support intersecting $\partial\Omega$. This will be done by a standard extension of g in a neighborhood of Ω as in [BO]. Nevertheless, in order to apply Lemma 3.5, we need a map φ which has compact support (say in $\check{B}_{2r}(x_0)$), hence, as in [BOS], we adapt the definition of the energy temporarily. We define a cut-off function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$

$$f(a,b) := \begin{cases} 1 & \text{if } b \leq a, \\ 2 - b/a & \text{if } a \leq b \leq 2a, \\ 0 & \text{if } 2a \leq b. \end{cases}$$

For $x_0 \in \bar{\Omega}$ and $r > 0$, we then set

$$\bar{E}_\varepsilon(x_0, r) := \frac{1}{r^{N-2}}\int_{\check{B}_{2r}(x_0)} e_\varepsilon(u) f(r, |x-x_0|)\ dx.$$

An integration by parts shows that, for any $F \geq 0$ measurable,

$$\int_{\check{B}_{2r}} F(x) f(r,|x|) = \int_1^2 \int_{\check{B}_{rt}} F(x)\ dt. \tag{41}$$

This formula is the link between the usual scaled energy $\tilde{E}_\varepsilon$ and $\bar{E}_\varepsilon$.

Lemma 3.6. *Assume u satisfies* (1)-(5), $x_0 \in \bar{\Omega}$ *and* $r > 0$. *Then,*

$$\begin{aligned}
\frac{d}{dr}(\bar{E}_\varepsilon(x_0, r)) = {} & \frac{1}{r^{N-2}} \int_{\partial\Omega \cap B_{2r}(x_0)} e_\varepsilon(u) f(r, |x - x_0|) + \frac{1}{r^{N-2}} \int_1^2 t \int_{\Omega \cap \partial B_{tr}(x_0)} \Big|\frac{\partial u}{\partial n}\Big|^2 \\
& + \frac{1}{r^{N-1}} \int_1^2 \int_{B_{tr}(x_0) \cap \partial\Omega} \frac{1}{2}(x - x_0) \cdot n \Big|\frac{\partial u}{\partial n}\Big|^2 \, dt \\
& + \frac{1}{r^{N-1}} \int_{\check{B}_{2r}(x_0)} \frac{(a_\varepsilon(x) - |u|^2)^2}{2\varepsilon^2} f(r, |x - x_0|) \\
& - \frac{N-1}{2r^{N-1}} |\log\varepsilon| \int_{\check{B}_{2r}(x_0)} \langle Ju, \sum_i c_i(x) \xi_i(x - x_0) f(r, |x - x_0|) \rangle \\
& - \frac{|\log\varepsilon|^2}{2r^{N-1}} \int_{\check{B}_{2r}(x_0)} (a_\varepsilon(x) - |u|^2)((x - x_0) \cdot \nabla d(x)) f(r, |x - x_0|) \\
& - \frac{1}{r^{N-1}} \int_{B_{2r}(x_0) \cap \partial\Omega} (x - x_0) \cdot n e_\varepsilon^\top(g_\varepsilon) f(r, |x - x_0|) \\
& + \frac{1}{r^{N-1}} \int_{B_{2r}(x_0) \cap \partial\Omega} (\frac{\partial u}{\partial n}, (x - x_0)_\top \cdot \nabla_\top g_\varepsilon) f(r, |x - x_0|).
\end{aligned} \tag{42}$$

Proof. We still assume $x_0 = 0$. First, one has

$$\begin{aligned}
\frac{d\bar{E}_\varepsilon}{dr} = {} & - \frac{N-2}{r^{N-1}} \int_{\check{B}_{2r}} e_\varepsilon(u) f(r, |x|) + \frac{1}{r^{N-2}} \int_{\Omega \cap \partial B_{2r}} e_\varepsilon(u) f(r, |x|) \\
& + \frac{1}{r^{N-2}} \int_{\check{B}_{2r}} e_\varepsilon(u) \partial_r f(r, |x|).
\end{aligned} \tag{43}$$

In (43), we have then for the first term (by (41))

$$-\frac{N-2}{r^{N-1}} \int_{\check{B}_{2r}} e_\varepsilon(u) f(r, |x|) = -\frac{N-2}{r^{N-1}} \int_1^2 \int_{\check{B}_{rt}} e_\varepsilon(u) \, dt$$

and for the third term

$$\frac{1}{r^{N-2}} \int_{\check{B}_{2r}} e_\varepsilon(u) \partial_r f(r, |x|) = \frac{1}{r^{N-2}} \int_r^{2r} \int_{\partial\Omega \cap B_\rho} e_\varepsilon(u) \frac{\rho}{r^2} d\rho = \frac{1}{r^{N-2}} \int_1^2 t \int_{\partial\Omega \cap B_{tr}} e_\varepsilon(u),$$

thus

$$\begin{aligned}
\frac{d\bar{E}_\varepsilon}{dr} = {} & \frac{1}{r^{N-2}} \int_{\partial\Omega \cap \check{B}_{2r}} e_\varepsilon(u) f(r, |x|) \\
& + \int_1^2 t^{N-1} \Big(-\frac{N-2}{(tr)^{N-1}} \int_{\check{B}_{rt}} e_\varepsilon(u) + \frac{1}{(rt)^{N-2}} \int_{\partial\Omega \cap B_{rt}} e_\varepsilon(u) \Big) \, dt.
\end{aligned}$$

The term between parenthesis is $\tilde{E}'_\varepsilon(rt)$, hence inserting formula (26) for rt and using formula (41) we are led to the conclusion. □

Lemma 3.7. (Monotonicity at large scales). *There exist constants $R > 0$, depending only on Ω, and C, depending only on Ω and ν, such that, for any $x_0 \in \bar{\Omega}$, $r_\varepsilon \leq r \leq \min(R, (1 + \Lambda_0)^{-1/\nu})$ and u solution to* (1)-(5), *we have for every $r_\varepsilon \leq s < r$,*

$$\bar{E}_\varepsilon(s) \leq C(\bar{E}_\varepsilon(r) + T_\varepsilon^\nu(s, 2r) + \Lambda_0 \varepsilon^\beta).$$

Proof. We assume $x_0 = 0$ and we estimate each term on the right-hand side of (42). For the sixth term, we have as for (34)

$$\Big|\frac{|\log\varepsilon|^2}{2r^{N-1}}\int_{\check{B}_r}(a_\varepsilon(x)-|u|^2)(x\cdot\nabla d(x))f(r,|x|)\Big| \le \tilde{E}_\varepsilon(r)+Cr^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4. \tag{44}$$

Concerning the fifth one, we proceed as in [BO] (Proposition 2.1 there). First, we extend u outside the domain. There exists $\delta_0>0$ such that the nearest point projection Π is well-defined and is a smooth fibration from the δ_0-neighborhood $(\partial\Omega)_{\delta_0}$ of $\partial\Omega$ onto $\partial\Omega$, inducing smooth diffeomorphisms $\Pi_t:\partial\Omega_t\to\partial\Omega$ $(0\le t\le\delta_0)$. We extend u in a map $\tilde{u}$ in Ω_{δ_0} by setting

$$\tilde{u}=u\circ\Pi \quad \text{ on } \ \Omega_{\delta_0}\setminus\Omega.$$

We extend in the same way the c_i's $(1\le i\le N)$ and d on Ω_{δ_0}. Finally, we extend the ξ_i's as in [BO], that is we write on $\partial\Omega$

$$\xi_i=(\xi_i)_\top+(\xi_i)_N,$$

where $(\xi_i)_\top$ and $(\xi_i)_N$ are respectively the tangential and the normal components of ξ_i on $\partial\Omega$ (see the Appendix of [BBO] for notations), and then we set, if $d(x,\partial\Omega)=t$,

$$\tilde{\xi}_i(x)=(\Pi_t^{-1})^*(\xi_i)_\top(x)+(\xi_i)_N(\Pi(x)),$$

where Π_t^{-1} denotes the inverse of the diffeomorphism Π_t. Next, we write

$$\frac{N-1}{2r^{N-1}}|\log\varepsilon|\cdot\Big|\int_{\check{B}_{2r}}\langle Ju,\sum_i c_i(x)\xi_i(x)f(r,|x|)\rangle\Big| \le \frac{C}{r^{N-1}}|\log\varepsilon|\Big(\Big|\int_{B_{2r}}\langle J\tilde{u},\varphi\rangle\Big|+\Big|\int_{B_{2r}\setminus\Omega}\langle J\tilde{u},\varphi\rangle\Big|\Big),$$

where

$$\varphi(x):=\sum_i\tilde{c}_i(x)\tilde{\xi}_i(x)f(r,|x|).$$

The first integral is estimated with Lemma 3.5. Since $\varphi\in\mathcal{C}_c^{0,1}(B_{2r},\Lambda^2\mathbb{R}^N)$ and

$$\|\varphi\|_\infty\le C\Lambda_0 r, \qquad \|d\varphi\|_\infty\le C\Lambda_0, \tag{45}$$

we obtain

$$\frac{|\log\varepsilon|}{r^{N-1}}\Big|\int_{B_{2r}}\langle J\tilde{u},\varphi\rangle\Big| \le C\Lambda_0\tilde{E}_\varepsilon(2r)+\frac{C}{r^{N-1}}\Lambda_0\varepsilon^\alpha|\log\varepsilon|(1+E_\varepsilon(2r)).$$

For the second integral, we have as in [BO], using the coarea formula and with $d=dist(.,\partial\Omega)$ (verifying $|\nabla d|=1$),

$$\begin{aligned}\Big|\int_{B_{2r}\setminus\Omega}\langle J\tilde{u},\varphi\rangle\Big| &= \Big|\int_{B_{2r}\setminus\Omega}|\nabla d|\langle\Pi^*Jg_\varepsilon,\varphi\rangle\Big|\\ &= \Big|\int_0^{2r}dt\int_{d^{-1}(t)}\sum_i f(r,|x|)c_i(x)\langle\Pi_t^*Jg_\varepsilon,(\Pi_t^{-1})^*(\xi_i)_\top\rangle\Big|\\ &\le C(\Omega)r\Big|\int_{B_{2r}\cap\partial\Omega}\langle Jg_\varepsilon,\varphi_\top\rangle\Big|.\end{aligned}$$

If $N\ge3$ (if $N=2$, $Jg_\varepsilon\equiv0$), to estimate the last integral, we also invoke Jerrard-Soner's result of Lemma 3.5 with this time the smooth manifold $B_{2r}\cap\partial\Omega$ of dimension $N-1\ge2$ and $\varphi_\top\in\mathcal{C}_c^{0,1}(B_{2r}\cap\partial\Omega,\Lambda^2\mathbb{R}^N)$ satisfying also (45), thus

$$\frac{|\log\varepsilon|}{r^{N-1}}\Big|\int_{B_{2r}\setminus\Omega}\langle J\tilde{u},\varphi\rangle\Big| \le C\Lambda_0 rG_\varepsilon(2r)+\frac{C}{r^{N-2}}\Lambda_0\varepsilon^\alpha|\log\varepsilon|(1+r^{N-2}G_\varepsilon(2r)).$$

We therefore deduce the estimate for the fifth term in (42) ($r \le 1$)

$$
\begin{aligned}
\frac{1}{r^{N-1}}|\log\varepsilon|\cdot\Big|\int_{\check{B}_{2r}}\langle Ju,\sum_i c_i(x)\xi_i(x)f(r,|x|)\rangle\Big|
&\le C\Lambda_0\tilde{E}_\varepsilon(2r)+\frac{C}{r^{N-1}}\Lambda_0\varepsilon^\alpha|\log\varepsilon|(1+E_\varepsilon(2r))\\
&\quad+C\Lambda_0 rG_\varepsilon(2r)+\frac{C}{r^{N-2}}\Lambda_0\varepsilon^\alpha|\log\varepsilon|(1+r^{N-2}G_\varepsilon(2r))\\
&\le C\Lambda_0(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\bar{E}_\varepsilon(4r)+C\Lambda_0 rG_\varepsilon(2r)(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})+C\Lambda_0\frac{\varepsilon^\alpha|\log\varepsilon|}{r^{N-1}}. \qquad (46)
\end{aligned}
$$

For the seventh term, we have clearly

$$
\frac{1}{r^{N-1}}\int_{B_{2r}\cap\partial\Omega}x\cdot ne_\varepsilon^\top(g_\varepsilon)f(r,|x|)\le 2^{N-1}G_\varepsilon(2r). \qquad (47)
$$

We estimate also the normal derivative as for (37), using estimates similar to (44) and (46),

$$
\begin{aligned}
\frac{1}{r^{N-3}}\int_{\partial\Omega\cap B_{2r}}\Big|\frac{\partial u}{\partial n}\Big|^2 &\le C\Big(1+\Lambda_0(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)\bar{E}_\varepsilon(4r) \qquad (48)\\
&\quad+C\Big(1+\Lambda_0 r(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)G_\varepsilon(2r)+C\Lambda_0^2r^2\varepsilon^2|\log\varepsilon|^4+C\Lambda_0\frac{\varepsilon^\alpha|\log\varepsilon|}{r^{N-1}}.
\end{aligned}
$$

We infer from (48) the estimate for the third term in (42) as for (38) (using (32))

$$
\begin{aligned}
\frac{1}{2r^{N-1}}\int_1^2\int_{B_{tr}(x_0)\cap\partial\Omega}x\cdot n\Big|\frac{\partial u}{\partial n}\Big|^2\,dt &\ge \frac{1}{2r^{N-1}}\int_{B_r\cap\partial\Omega}|x\cdot n|\cdot\Big|\frac{\partial u}{\partial n}\Big|^2\\
&\quad-C\Big(1+\Lambda_0(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)\bar{E}_\varepsilon(4r)-C\Big(1+r\Lambda_0(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)G_\varepsilon(2r)\\
&\quad-\Lambda_0^2\varepsilon^2|\log\varepsilon|^4-\Lambda_0\frac{\varepsilon^\alpha|\log\varepsilon|}{r^{N-1}}. \qquad (49)
\end{aligned}
$$

For the last term in (42), we obtain as for (39) and using (48)

$$
\begin{aligned}
\Big|\frac{1}{r^{N-1}}\int_{B_{2r}\cap\partial\Omega}(\frac{\partial u}{\partial n},x_\top\cdot\nabla_\top g_\varepsilon)f(r,|x|)\Big|
&\le C\Big(1+\Lambda_0(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)\frac{\nu\bar{E}_\varepsilon(4r)}{(4r)^{1-\nu}}+C\Big(1+\Lambda_0 r(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)\frac{G_\varepsilon(2r)}{r^\nu} \qquad (50)\\
&\quad+C\Lambda_0^2\varepsilon^2|\log\varepsilon|^4r^{2-\nu}+C\Lambda_0\frac{\varepsilon^\alpha|\log\varepsilon|}{r^{N-1+\nu}}.
\end{aligned}
$$

Combining estimates (44), (46), (47), (49) and (50) with (42) yields

$$
\begin{aligned}
\frac{d\bar{E}_\varepsilon}{dr} &\ge \frac{1}{r^{N-2}}\int_1^2 t\int_{\Omega\cap\partial B_{tr}}\Big|\frac{\partial u}{\partial n}\Big|^2+\frac{1}{r^{N-1}}\int_1^2\int_{B_{tr}\cap\partial\Omega}\frac{|x\cdot n|}{2}\Big|\frac{\partial u}{\partial n}\Big|^2\,dt\\
&\quad+\frac{1}{r^{N-1}}\int_{\check{B}_{2r}}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2}f(r,|x|)-C\Big(1+\Lambda_0(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)\frac{\nu\bar{E}_\varepsilon(4r)}{(4r)^{1-\nu}} \qquad (51)\\
&\quad-C\Big(1+\Lambda_0 r(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\Big)\frac{G_\varepsilon(2r)}{r^\nu}-C\Lambda_0^2\varepsilon^2|\log\varepsilon|^4r^{2-\nu}-C\Lambda_0\frac{\varepsilon^\alpha|\log\varepsilon|}{r^{N-1+\nu}}.
\end{aligned}
$$

Now, we assume r large enough so that $r^{1-\nu-N}\varepsilon^\alpha|\log\varepsilon| \geq r^{1-N}\varepsilon^\alpha|\log\varepsilon|$ (hence $r^{-1}\varepsilon^\alpha|\log\varepsilon|$) tends to zero with ε. We therefore assume

$$r \geq (\varepsilon^{-\beta}\varepsilon^\alpha|\log\varepsilon|)^{1/(N-1)} =: r_\varepsilon, \tag{52}$$

where $0 < \beta < \alpha$ is fixed (for instance, $\beta = \alpha/2$, and set $\mu := \alpha - \beta > 0$), so that (51) implies, for $r_\varepsilon \leq r \leq \min(R, (1+\Lambda_0)^{-1/\nu})$

$$\frac{d\bar{E}_\varepsilon}{dr} \geq A(r) - C\Big((1+\Lambda_0)\frac{\bar{E}_\varepsilon(4r)}{(4r)^{1-\nu}} + \frac{G_\varepsilon(2r)}{r^\nu} + \Lambda_0\varepsilon^\beta\Big), \tag{53}$$

with

$$\begin{aligned} A(r) := {} & \frac{1}{r^{N-2}}\int_1^2 t\int_{\Omega\cap\partial B_{tr}}\Big|\frac{\partial u}{\partial n}\Big|^2 + \frac{1}{r^{N-1}}\int_1^2\int_{B_{tr}\cap\partial\Omega}\frac{|x\cdot n|}{2}\Big|\frac{\partial u}{\partial n}\Big|^2\,dt \\ & + \frac{1}{r^{N-1}}\int_{\check{B}_{2r}}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2}f(r,|x|). \end{aligned} \tag{54}$$

In particular, since $A(r) \geq 0$, for $r \geq r_\varepsilon$,

$$\frac{d\bar{E}_\varepsilon}{dr} \geq -C\Big((1+\Lambda_0)\frac{\bar{E}_\varepsilon(4r)}{(4r)^{1-\nu}} + \frac{G_\varepsilon(2r)}{r^\nu} + \Lambda_0\varepsilon^\beta\Big). \tag{55}$$

To conclude the proof, we will need the following discrete Gronwall inequality.

Lemma 3.8. (Discrete Gronwall inequality). *Let $0 < s_1 < 4s_1 < s_2 \leq 1$, $f : [s_1, s_2] \to \mathbb{R}_+$ be continuous and assume $h : [s_1, s_2] \to \mathbb{R}_+$ is continuously differentiable and satisfies*

$$\begin{cases} h(s) \leq \theta^{N-2}h(\theta s) & \text{if } \theta \in [1, s_2/s_1],\ s \in [s_1, s_2] \text{ and } \theta s \in [s_1, s_2], \\ h'(s) \geq -C\frac{h(4s)}{(4s)^{1-\nu}} - f(s) & \text{for all } s \in [s_1, s_2/4], \end{cases} \tag{56}$$

for constants $C > 0$ and $\nu \in (0, 1]$. Then, for all $s_1 \leq s < t \leq s_2$,

$$h(s) \leq 4^{N-2}\exp(C\lambda t^\nu)h(t) + \Big(\int_s^{t/4} f(r)\,dr\Big)\exp(C\lambda t^\nu), \tag{57}$$

where

$$\lambda := \frac{1}{4^\nu(4^\nu - 1)}.$$

Proof. We proceed by induction. Let $s_1 \leq s < t < s_2$. Assume $t/4 \leq s \leq t$. Then, by (56),

$$h(s) \leq 4^{N-2}h(t).$$

Assume that for some $k \in \mathbb{N}^*$, it holds

$$h(s) \leq 4^{N-2}h(t)\prod_{i=1}^{k-1}(1 + \frac{3Ct^\nu}{4^{\nu i+1}}) + \int_s^{t/4^{k-1}} f(\rho)\,d\rho + \sum_{j=2}^{k-1}(\int_{t/4^j}^{t/4^{j-1}} f(\rho)\,d\rho)\prod_{i=j-1}^{k-1}(1 + \frac{3Ct^\nu}{4^{1+\nu i}})$$

for all $\frac{t}{4^k} \leq s \leq \frac{t}{4^{k-1}}$. If $\frac{t}{4^{k+1}} \leq s \leq \frac{t}{4^k}$, then, by (56) and using the fact that

$$\int_s^{t/4^k}\frac{1}{(4r)^{1-\nu}}\,dr \leq (\frac{t}{4^k} - s)\frac{1}{(t/4^k)^{1-\nu}} \leq \frac{3t}{4^{k+1}}\frac{4^{(1-\nu)k}}{t^{1-\nu}} = \frac{3t^\nu}{4^{k\nu+1}},$$

we obtain

$$
\begin{aligned}
h(s) &\le h(t/4^k) + C\int_s^{t/4^k} \frac{h(4r)}{(4r)^{1-\nu}}\, dr + \int_s^{t/4^k} f(\rho)\, d\rho \\
&\le \int_s^{t/4^k} f(\rho)\, d\rho + 4^{N-2}h(t)\prod_{i=1}^{k-1}(1+\frac{3Ct^\nu}{4^{\nu i+1}}) + \int_{t/4^k}^{t/4^{k-1}} f(\rho)\, d\rho \\
&\quad + \sum_{j=2}^{k-1}\Big((\int_{t/4^j}^{t/4^{j-1}} f(\rho)\, d\rho) \prod_{i=j-1}^{k-1}(1+\frac{3Ct^\nu}{4^{1+\nu i}})\Big) + \frac{3Ct^\nu}{4^{\nu k+1}}4^{N-2}h(t)\prod_{i=1}^{k-1}(1+\frac{3Ct^\nu}{4^{\nu i+1}}) \\
&\quad + \frac{3Ct^\nu}{4^{\nu k+1}}\Big(\int_{t/4^k}^{t/4^{k-1}} f(\rho)\, d\rho + \sum_{j=2}^{k-1}\Big((\int_{t/4^j}^{t/4^{j-1}} f(\rho)\, d\rho) \prod_{i=j-1}^{k-1}(1+\frac{3Ct^\nu}{4^{1+\nu i}})\Big) \\
&\le 4^{N-2}h(t)\prod_{i=1}^{k}(1+\frac{3Ct^\nu}{4^{\nu i+1}}) + \int_s^{t/4^k} f(\rho)\, d\rho + (1+\frac{3Ct^\nu}{4^{k\nu+1}})\int_{t/4^k}^{t/4^{k-1}} f(\rho)\, d\rho \\
&\quad + \sum_{j=2}^{k-1}\Big((\int_{t/4^j}^{t/4^{j-1}} f(\rho)\, d\rho)(1+\frac{3Ct^\nu}{4^{k\nu+1}}) \prod_{i=j-1}^{k-1}(1+\frac{3Ct^\nu}{4^{1+\nu i}})\Big) \\
&\le 4^{N-2}h(t)\prod_{i=1}^{k}(1+\frac{3Ct^\nu}{4^{\nu i+1}}) + \int_s^{t/4^k} f(\rho)\, d\rho + \sum_{j=2}^{k}\Big((\int_{t/4^j}^{t/4^{j-1}} f(\rho)\, d\rho) \prod_{i=j-1}^{k}(1+\frac{3Ct^\nu}{4^{1+\nu i}})\Big).
\end{aligned}
$$

The conclusion follows then from the inequality, valid for all $m \in \mathbb{N}$,

$$
\prod_{i=1}^{m}(1+\frac{3Ct^\nu}{4^{\nu i+1}}) \le \exp\Big(\frac{3}{4}Ct^\nu\sum_{i=1}^{\infty}4^{-\nu i}\Big) \le \exp(C\lambda t^\nu),
$$

by definition of λ. Indeed, we have

$$
\begin{aligned}
h(s) &\le 4^{N-2}h(t)\exp(C\lambda t^\nu) + \int_s^{t/4^k} f(\rho)\, d\rho + \exp(C\lambda t^\nu)\sum_{j=1}^{k}\int_{t/4^j}^{t/4^{j-1}} f(\rho)\, d\rho \\
&\le 4^{N-2}h(t)\exp(C\lambda t^\nu) + \exp(C\lambda t^\nu)\int_s^{t/4} f(\rho)\, d\rho
\end{aligned}
$$

and the proof is complete. □

To conclude the proof of Lemma 3.7, we apply Lemma 3.8 with $s_1 = r_\varepsilon$, $s_2 = r$, $h = \bar{E}_\varepsilon$, $f(r) = C(r^{-\nu}G_\varepsilon(2r) + \Lambda_0\varepsilon^\beta)$, $C = C(1+\Lambda_0)$. Note that $\lambda s_2^\nu = (1+\Lambda_0)r^\nu \le 1$. The first hypothesis in (56) is easily verified for the modified scaled energy $\bar{E}_\varepsilon$ and the second one is (55). We then infer that, for every $r_\varepsilon \le s < r \le \min(R, (1+\Lambda_0)^{-1/\nu})$,

$$
\bar{E}_\varepsilon(s) \le C(\bar{E}_\varepsilon(r) + T_\varepsilon^\nu(s, 2r) + \Lambda_0\varepsilon^\beta).
$$

This finishes the proof of Lemma 3.7. □

3.2 The Neumann problem

In this subsection, we point out the modifications to make in order to handle the Neumann case, that is for solutions u to (1)-(6). In the sequel, C is a constant depending only on Ω (and N).

Lemma 3.9. *Let u be a solution of* (1)-(6) *in Ω, then for $r > 0$ and $x_0 \in \bar{\Omega}$*

$$\begin{aligned}\frac{d}{dr}(\tilde{E}_\varepsilon(x_0, r)) = {} & \frac{1}{r^{N-2}} \int_{\Omega \cap \partial B_r(x_0)} \left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{r^{N-1}} \int_{\check{B}_r(x_0)} \frac{(a_\varepsilon(x) - |u|^2)^2}{2\varepsilon^2} \\ & - \frac{N-1}{2r^{N-1}} |\log \varepsilon| \int_{\check{B}_r(x_0)} \langle Ju, \sum_i c_i(x) \xi_i(x - x_0)\rangle \\ & - \frac{|\log \varepsilon|^2}{2r^{N-1}} \int_{\check{B}_r(x_0)} (a_\varepsilon(x) - |u|^2)(x - x_0) \cdot \nabla d(x) \\ & - \frac{1}{r^{N-1}} \int_{B_r(x_0) \cap \partial\Omega} (x - x_0) \cdot n e_\varepsilon(u).\end{aligned} \tag{58}$$

Proof. Assuming $x_0 = 0$, we still have formula (27). It suffices to use the Neumann condition (6) to obtain (58), since the last term in the last integral in (27) is 0. $\square$

This time, the last term in equality (58) involves the energy on the boundary of u. Note that this term is not so bad since the term $(x - x_0) \cdot n$ is expected to be of order r^2 if x_0 is close enough to the boundary. The next lemma, analoguous to Lemma 3.3, provides an estimate for this term.

Lemma 3.10. (Control of the boundary energy). *Let u be a solution of* (1)-(6). *There exist C and $0 < R \leq 1$ depending only on Ω such that, for all $x_0 \in \bar{\Omega}$ and $0 < r < R$, there exists $z_0 \in \check{B}_r(x_0)$ such that*

$$\begin{aligned} r \int_{\partial\Omega \cap B_r(x_0)} e_\varepsilon(u) \leq C\Big(\int_{\check{B}_r(x_0)} e_\varepsilon(u) + |\log \varepsilon| \cdot \Big|\int_\omega \langle Ju, \sum_i c_i(x) \xi_i(x - z_0)\rangle\Big| \\ + \frac{|\log \varepsilon|^2}{2} \Big|\int_\omega (a_\varepsilon - |u|^2)(x - z_0) \cdot \nabla d\Big|\Big),\end{aligned} \tag{59}$$

where $\omega \subset \check{B}_r(x_0)$ depends on u, x_0 and r.

Proof. The proof begins as for Lemma 3.3, that is assuming first that $\partial\Omega$ is locally the half-plane $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{0\}$, that $a \leq r/4$ and exhibiting by averaging y and $r' \in (r, 9r/8)$ such that

$$\int_{\mathbb{R}^N_+ \cap \partial B_{r'}(y)} e_\varepsilon(u) \leq \frac{C}{r} \int_{\check{B}_r(x_0)} e_\varepsilon(u). \tag{60}$$

We also have for an $\alpha > 0$,

$$(x - z_0) \cdot n \geq \alpha r. \tag{61}$$

We also apply the Pohozaev identity of Lemma 3.1 with z_0, x_0 and ω and use the Neumann condition (6) to obtain

$$\begin{aligned} C \int_{\check{B}_r(x_0)} e_\varepsilon(u) + C|\log \varepsilon| \cdot \Big|\int_\omega \langle Ju, \sum_i c_i(x) \xi_i(x - z_0)\rangle\Big| \\ + |\log \varepsilon|^2 \Big|\int_\omega (a_\varepsilon(x) - |u|^2)(x - z_0) \cdot \nabla d(x)\Big| \geq 2 \int_{\partial\Omega \cap B_r} (x - z_0) \cdot n e_\varepsilon(u) - Cr \int_{\Omega \cap \partial B_{r'}(y)} e_\varepsilon(u).\end{aligned}$$

The last integral is estimated by (60) and for the before last integral, we use the starshapedness assumption (61) to obtain

$$C\int_{\check{B}_r(x_0)} e_\varepsilon(u) + C|\log\varepsilon|\cdot\Big|\int_\omega \langle Ju, \sum_i c_i(x)\xi_i(x-z_0)\rangle\Big|$$
$$+ C|\log\varepsilon|^2\Big|\int_\omega (a_\varepsilon(x)-|u|^2)(x-z_0)\cdot\nabla d(x)\Big| \geq \alpha r\int_{\partial\Omega\cap B_r(x_0)} e_\varepsilon(u)$$

and the conclusion follows. If $a \geq r/4$ or for a general domain Ω, the proof is the same as for Lemma 3.3. $\square$

The monotonicity formula for small scales is then given in the following lemma, where χ stands for the characteristic function.

Lemma 3.11. (Monotonicity at small scales). *There exist C and $0 < R \leq 1$, depending only on N and Ω, such that for any solution u of (1)-(6) in Ω any $x_0 \in \bar{\Omega}$ and $0 < r \leq \min(R, \Lambda^{-1})$, with*

$$d_0 := dist(x_0, \partial\Omega), \qquad \Lambda := C(1+\Lambda_0|\log\varepsilon|), \qquad Q := C\Lambda_0\varepsilon|\log\varepsilon|^2,$$

we have, with the convention $d_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r}) = 0$ if $d_0 = 0$,

$$\frac{d}{dr}\Big(\exp\Big[\Lambda r + C d_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r})\Big](\tilde{E}_\varepsilon + \frac{Q^2}{\Lambda})\Big)$$
$$\geq \frac{1}{r^{N-1}}\int_{\Omega\cap\partial B_r(x_0)} \frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} + \frac{1}{r^{N-2}}\int_{\partial B_r(x_0)\cap\Omega}\Big|\frac{\partial u}{\partial n}\Big|^2 \geq 0.$$

In particular, $\exp[\Lambda r + C d_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r})](\tilde{E}_\varepsilon + \frac{Q^2}{\Lambda})$ is a nondecreasing function on $(0,R)$.

Proof. First, we note that, if $0 < r \leq R(\Omega)$ sufficiently small, for all $x \in \check{B}_r(x_0)$,

$$(x-x_0)\cdot n \leq C(d_0+r^2), \tag{62}$$

where C depends on Ω only. This is a basic difference with Lemma 3.4. Arguing as in Lemma 3.4, that is assuming that for $r < R$ sufficiently small, $\check{B}_r(x_0)$ is the uppergraph of a map $\psi : B_1(0) \subset \mathbb{R}^{N-1} \to \mathbb{R}$ and that $\psi(0) = |\nabla\psi(0)| = 0$, so that the tangent hyperplane at Ω at $\psi(0) = 0$ is $\mathbb{R}^{N-1}\times\{0\}$, we are led to prove, as for the proof of (32), that for $x \in \check{B}_r(x_0)$,

$$(x-x_0)\cdot n = (x-x_0)\cdot\Big(\vec{e}_N - \sum_{i=1}^{N-1}\partial_i\psi\vec{e}_i\Big) \leq C(d_0+r^2).$$

It is clear that

$$(x-x_0)\cdot\vec{e}_N \leq C(d_0+r^2) \tag{63}$$

since either d_0 is of greater than or of the order of r and then inequality (63) is true, either $d_0 \ll r \leq R$ and then inequality (63) is also true. Since $|\nabla\psi| \leq Cr$ and $|x-x_0| \leq r$, the second term in (62) is $\leq Cr^2$. Therefore, (62) holds. We now turn to the proof of Lemma 3.11.
We proceed as in Lemma 3.4 and estimate each term on the right hand side of (58). The third

one and the fourth one are treated as in (33) (using the rough estimate for the jacobian) and (34), which yields respectively

$$\Big|\frac{N-1}{2r^{N-1}}|\log\varepsilon|\int_{\check{B}_r(x_0)}\langle Ju,\sum_i c_i(x)\xi_i(x)\rangle\Big| \leq C\Lambda_0|\log\varepsilon|\tilde{E}_\varepsilon(r) \tag{64}$$

and

$$\Big|\frac{|\log\varepsilon|^2}{2r^{N-1}}\int_{\check{B}_r}(a_\varepsilon(x)-|u|^2)x\cdot\nabla d(x)\Big| \leq \tilde{E}_\varepsilon(r)+C\Lambda_0^2\varepsilon^2|\log\varepsilon|^4. \tag{65}$$

We use (59) of Lemma 3.10 to estimate the last term in (58)

$$\begin{aligned}r\int_{\partial\Omega\cap B_r}e_\varepsilon(u) \leq C\Big(\int_{\check{B}_r}e_\varepsilon(u)+|\log\varepsilon|\cdot\Big|\int_\omega\langle Ju,\sum_i c_i(x)\xi_i(x-z_0)\rangle\Big| \\ +\frac{|\log\varepsilon|^2}{2}\Big|\int_\omega(a_\varepsilon-|u|^2)(x-z_0)\cdot\nabla d\Big|\Big).\end{aligned} \tag{66}$$

Note that we do not need to estimate the last term in (58) if $r<d_0$, since in this case, $\check{B}_r=\emptyset$. Therefore, (62) and (66) imply

$$\begin{aligned}\frac{1}{r^{N-1}}\int_{\partial\Omega\cap B_r}x\cdot ne_\varepsilon(u) \leq C\chi_{\{r\geq d_0\}}\frac{d_0+r^2}{r^N}\Big(\int_{\check{B}_r}e_\varepsilon(u)+|\log\varepsilon|\cdot\Big|\int_\omega\langle Ju,\sum_i c_i(x)\xi_i(x-z_0)\rangle\Big| \\ +\frac{|\log\varepsilon|^2}{2}\Big|\int_\omega(a_\varepsilon-|u|^2)(x-z_0)\cdot\nabla d\Big|\Big).\end{aligned}$$

We estimate the two last terms as in (64) and (65) to infer

$$\frac{1}{r^{N-1}}\int_{\partial\Omega\cap B_r}x\cdot ne_\varepsilon(u) \leq C\chi_{\{r\geq d_0\}}(1+\frac{d_0}{r^2})\Big((1+r\Lambda_0|\log\varepsilon|)\tilde{E}_\varepsilon(r)+\Lambda_0^2\varepsilon^2|\log\varepsilon|^4\Big). \tag{67}$$

Inserting estimates (64), (65) and (67) into (58) gives, with $\Lambda=C(1+\Lambda_0|\log\varepsilon|)$ and for $r\leq\Lambda^{-1}$,

$$\begin{aligned}\frac{d\tilde{E}_\varepsilon}{dr} &\geq \frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r}\Big|\frac{\partial u}{\partial n}\Big|^2+\frac{1}{r^{N-1}}\int_{\check{B}_r}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} \\ &\quad -(\Lambda+C\chi_{\{r\geq d_0\}}\frac{d_0}{r^2})\tilde{E}_\varepsilon(r)-C(1+\chi_{\{r\geq d_0\}}\frac{d_0}{r^2})\Lambda_0^2\varepsilon^2|\log\varepsilon|^4.\end{aligned}$$

To conclude, we introduce the primitive

$$\int_0^r\chi_{\{\rho\geq d_0\}}\frac{d_0}{\rho^2}\,d\rho=d_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r})\geq 0.$$

Consequently,

$$\begin{aligned}&\frac{d}{dr}\Big(\exp\Big[\Lambda r+Cd_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r})\Big]\tilde{E}_\varepsilon\Big) \\ &\qquad=\exp\Big[\Lambda r+Cd_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r})\Big]\Big(\frac{d\tilde{E}_\varepsilon}{dr}+(\Lambda+Cd_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r}))\tilde{E}_\varepsilon(r)\Big) \\ &\qquad\geq \frac{1}{r^{N-2}}\int_{\Omega\cap\partial B_r}\Big|\frac{\partial u}{\partial n}\Big|^2+\frac{1}{r^{N-1}}\int_{\check{B}_r}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2} \\ &\qquad\quad -C(1+\chi_{\{r\geq d_0\}}\frac{d_0}{r^2})\exp\Big[\Lambda r+Cd_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r})\Big]\Lambda_0^2\varepsilon^2|\log\varepsilon|^4.\end{aligned}$$

To conclude the proof, just note that the last term is ($\Lambda \geq C$)

$$\geq -\frac{d}{dr}\Big(\frac{Q^2}{\Lambda}\exp\Big[\Lambda r + Cd_0\chi_{\{r\geq d_0\}}(\frac{1}{d_0}-\frac{1}{r})\Big]\Big),$$

where $Q = C\Lambda_0\varepsilon|\log\varepsilon|^2$. $\square$

In the next lemma, we compute the derivative of the modified scaled energy $\bar{E}_\varepsilon(x_0, r)$ in the same way as for Lemma 3.6.

Lemma 3.12. *Assume u satisfies* (1)-(6)*, $x_0 \in \bar{\Omega}$ and $r > 0$. Then,*

$$\begin{aligned}
\frac{d}{dr}(\bar{E}_\varepsilon(x_0,r)) = {} & \frac{1}{r^{N-2}}\int_{\partial\Omega\cap B_{2r}(x_0)} e_\varepsilon(u)f(r,|x-x_0|) + \frac{1}{r^{N-2}}\int_1^2 t\int_{\Omega\cap\partial B_{tr}(x_0)}\Big|\frac{\partial u}{\partial n}\Big|^2 \\
& + \frac{1}{r^{N-1}}\int_{\check{B}_{2r}(x_0)}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2}f(r,|x-x_0|) \\
& - \frac{N-1}{2r^{N-1}}|\log\varepsilon|\int_{\check{B}_{2r}(x_0)}\langle Ju, \sum_i c_i(x)\xi_i(x-x_0)f(r,|x-x_0|)\rangle \qquad (68)\\
& - \frac{|\log\varepsilon|^2}{2r^{N-1}}\int_{\check{B}_{2r}(x_0)}(a_\varepsilon(x)-|u|^2)((x-x_0)\cdot\nabla d(x))f(r,|x-x_0|) \\
& - \frac{1}{r^{N-1}}\int_{B_{2r}(x_0)\cap\partial\Omega}(x-x_0)\cdot n e_\varepsilon(u)f(r,|x-x_0|).
\end{aligned}$$

Lemma 3.13. (Monotonicity at large scales). *There exist constants C and $0 < R \leq 1$, depending only on Ω such that, if $x_0 \in \bar{\Omega}$, $r_\varepsilon \leq r \leq \min(R, (1+\Lambda_0)^{-1})$ and u is a solution to* (1)-(6)*, then for every $r_\varepsilon \leq s < r$,*

$$\bar{E}_\varepsilon(s) \leq C\exp\Big(C((1+\Lambda_0)\frac{t}{4} + 2d_0\chi_{\{t\geq 2d_0\}}(\frac{1}{2d_0}-\frac{1}{t}))\Big)(\bar{E}_\varepsilon(r)+\Lambda_0\varepsilon^\beta).$$

Proof. We assume $x_0 = 0$ and we estimate each term on the right-hand side of (68). For the fifth term, we have as for (65)

$$\Big|\frac{|\log\varepsilon|^2}{2r^{N-1}}\int_{\check{B}_{2r}}(a_\varepsilon(x)-|u|^2)(x\cdot\nabla d(x))f(r,|x|)\Big| \leq \tilde{E}_\varepsilon(r) + C\Lambda_0^2\varepsilon^2|\log\varepsilon|^4. \qquad (69)$$

Concerning the fourth one, we may use a reflection with respect to the boundary. We assume $2r \leq \delta_0$. We extend u in a map $\tilde{u}$ defined on $U := \check{B}_{2r}\cup\phi^{-1}(\check{B}_{2r})$ by setting for $x = \phi^{-1}(y) \in \phi^{-1}(\check{B}_{2r})$ ($y \in \check{B}_{2r}$),

$$\tilde{u}(x) := u(y) = u\circ\phi(x).$$

It is then clear that on $\phi^{-1}(\check{B}_{2r})$,

$$J\tilde{u} = \phi^* Ju$$

and that

$$E_\varepsilon(\tilde{u}, U) \leq CE_\varepsilon(u, \check{B}_{2r}).$$

We also extend the 2-form $\sum_i c_i(x)\xi_i(x)$ by this way setting $\varphi := \phi^*(\sum_i c_i(x)\xi_i(x))$ in $\phi^{-1}(\check{B}_{2r})$. This 2-form is in $\mathcal{C}_0^{0,1}(U, \Lambda^2\mathbb{R}^N)$ and satisfies

$$\|\varphi\|_\infty \leq C\Lambda_0 r \qquad \text{and} \qquad \|d\varphi\|_\infty \leq C\Lambda_0. \qquad (70)$$

Moreover,

$$\int_U \langle J\tilde{u}, \varphi(x)\rangle = 2\int_{\check{B}_{2r}} \langle Ju, \sum_i c_i(x)\xi_i(x)f(r,|x|)\rangle.$$

We apply the result of Jerrard and Soner of Lemma 3.5 for the first integral to infer

$$\Big|\int_{\check{B}_{2r}} \langle Ju, \sum_i c_i(x)\xi_i(x)f(r,|x|)\rangle\Big| \le C\Lambda_0 r\frac{E_\varepsilon(2r)}{|\log\varepsilon|} + C\Lambda_0\varepsilon^\alpha(1+E_\varepsilon(2r)).$$

Consequently, we have the following estimate for the fourth term in (68)

$$\frac{|\log\varepsilon|}{r^{N-1}}\Big|\int_{\check{B}_{2r}} \langle Ju, \sum_i c_i(x)\xi_i(x)f(r,|x|)\rangle\Big| \le C\Lambda_0(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r})\tilde{E}_\varepsilon(2r) + \frac{C\Lambda_0\varepsilon^\alpha|\log\varepsilon|}{r^{N-1}}. \tag{71}$$

We estimate also the boundary energy as for (67). We first apply Lemma 3.10 to obtain

$$\frac{1}{r^{N-3}}\int_{\partial\Omega\cap B_{2r}} e_\varepsilon(u) \le C\Big(\tilde{E}_\varepsilon(2r) + \frac{|\log\varepsilon|}{r^{N-2}}\Big|\int_\omega \langle Ju, \sum_i c_i(x)\xi_i(x)f(r,|x|)\rangle\Big|$$
$$+\frac{|\log\varepsilon|^2}{r^{N-2}}\Big|\int_\omega (a_\varepsilon(x)-|u|^2)(x\cdot\nabla d(x))f(r,|x|)\Big|\Big).$$

Using then (62), $\tilde{E}_\varepsilon(2r)\le \bar{E}_\varepsilon(4r)$, $r\le 1$ and estimates similar to (69) and (71), we infer for $r\le \min(R,(1+\Lambda_0)^{-1})$ the estimate of the last term in (68) as in (67)

$$\frac{1}{r^{N-1}}\int_{\partial\Omega\cap B_{2r}} x\cdot n e_\varepsilon(u)f(r,|x|)$$
$$\le C\Big(1+\chi_{\{2r\ge d_0\}}\frac{d_0}{4r^2}\Big)\Big(\bar{E}_\varepsilon(4r)+\Lambda_0^2\varepsilon^2|\log\varepsilon|^4+\Lambda_0\frac{\varepsilon^\alpha|\log\varepsilon|}{r^{N-2}}\Big). \tag{72}$$

Combining estimates (69), (71) and (72) with (68) yields

$$\frac{d\bar{E}_\varepsilon}{dr} \ge \frac{1}{r^{N-2}}\int_1^2 t\int_{\Omega\cap\partial B_{tr}}\Big|\frac{\partial u}{\partial n}\Big|^2 + \frac{1}{r^{N-1}}\int_{\check{B}_{2r}}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2}f(r,|x|) \tag{73}$$
$$-C\Big((1+\chi_{\{2r\ge d_0\}}\frac{d_0}{4r^2}+(1+\Lambda_0)(1+\frac{\varepsilon^\alpha|\log\varepsilon|}{r}))\bar{E}_\varepsilon(4r)+\Lambda_0^2\varepsilon^2|\log\varepsilon|^4+\Lambda_0\frac{\varepsilon^\alpha|\log\varepsilon|}{r^{N-1}}\Big).$$

As in Lemma 3.7, we assume

$$r\ge (\varepsilon^{-\beta}\varepsilon^\alpha|\log\varepsilon|)^{1/(N-1)} = r_\varepsilon, \tag{74}$$

where $0<\beta<\alpha$ is fixed (and take $\mu=\alpha-\beta>0$) so that $r^{-1}\varepsilon^\alpha|\log\varepsilon|\le r^{1-N}\varepsilon^\alpha|\log\varepsilon|\le\varepsilon^\beta$. Hence, with

$$B(r) := \frac{1}{r^{N-2}}\int_1^2 t\int_{\Omega\cap\partial B_{tr}}\Big|\frac{\partial u}{\partial n}\Big|^2 + \frac{1}{r^{N-1}}\int_{\check{B}_{2r}}\frac{(a_\varepsilon(x)-|u|^2)^2}{2\varepsilon^2}f(r,|x|), \tag{75}$$

(73) implies

$$\frac{d\bar{E}_\varepsilon}{dr} \ge B(r) - C\Big(1+\Lambda_0+\chi_{\{2r\ge d_0\}}\frac{d_0}{4r^2}\Big)\Big(\bar{E}_\varepsilon(4r)+\Lambda_0\varepsilon^\beta\Big). \tag{76}$$

In particular, since $B(r)\ge 0$, for $r_\varepsilon\le r\le\min(R,(1+\Lambda_0)^{-1})$,

$$\frac{d\bar{E}_\varepsilon}{dr} \ge -C\Big(1+\Lambda_0+\chi_{\{2r\ge d_0\}}\frac{d_0}{4r^2}\Big)\Big(\bar{E}_\varepsilon(4r)+\Lambda_0\varepsilon^\beta\Big). \tag{77}$$

To conclude the proof, we also make use of a discrete Gronwall inequality.

Lemma 3.14. (Discrete Gronwall inequality). *Let $0 < s_1 < 4s_1 < s_2$ and $h : [s_1, s_2] \to \mathbb{R}_+$ be continuously differentiable and such that*

$$\begin{cases} h(s) \leq \theta^{N-2} h(\theta s) & \text{if } \theta \in [1, s_2/s_1],\ s \in [s_1, s_2] \text{ and } \theta s \in [s_1, s_2], \\ h'(s) \geq -C(1 + \Lambda_0 + \chi_{\{2r \geq d_0\}} \frac{d_0}{4r^2})(h(4s) + D) & \text{for all } s \in [s_1, s_2/4], \end{cases}$$

where C, D and Λ_0 are positive constants. Then, for all $s_1 \leq s < t \leq s_2$,

$$h(s) \leq 4^{N-2} \exp\Big(C((1 + \Lambda_0)t/4 + 2d_0 \chi_{\{t \geq 2d_0\}}(\frac{1}{2d_0} - \frac{1}{t}))\Big)(h(t) + D). \tag{78}$$

Proof. We reduce the proof to the case $D = 0$ considering $g(s) := h(s) + D$. We have

$$g(s) = h(s) + D \leq \theta^{N-2} h(\theta s) + \theta^{N-2} D = \theta^{N-2} g(\theta s) \tag{79}$$

if $\theta \in [1, s_2/s_1]$, $s \in [s_1, s_2]$ and $\theta s \in [s_1, s_2]$ and

$$g'(s) = h'(s) \geq -C\Big(1 + \Lambda_0 + \chi_{\{2s \geq d_0\}} \frac{d_0}{4s^2}\Big)(h(4s) + D) = -C\Big(1 + \Lambda_0 + \chi_{\{2s \geq d_0\}} \frac{d_0}{4s^2}\Big) g(4s) \tag{80}$$

for all $s \in [s_1, s_2/4]$. Let $s_1 \leq s < t < s_2$. Assume $t/4 \leq s \leq t$. Then, by (79),

$$g(s) \leq 4^{N-2} g(t).$$

By induction, assume that for some $k \in \mathbb{N}^*$ it holds

$$g(s) \leq 4^{N-2} g(t) \prod_{i=2}^{k} (1 + C\alpha_i(t))$$

for all $\frac{t}{4^k} \leq s \leq \frac{t}{4^{k-1}}$, where we have set

$$\alpha_i(t) := \int_{t/4^i}^{t/4^{i-1}} (1 + \Lambda_0 + \chi_{\{2r \geq d_0\}} \frac{d_0}{4r^2})\, dr.$$

If $\frac{t}{4^{k+1}} \leq s \leq \frac{t}{4^k}$, then, by (80),

$$\begin{aligned} g(s) &\leq g(t/4^k) + C \int_s^{t/4^k} (1 + \Lambda_0 + \chi_{\{2r \geq d_0\}} \frac{d_0}{4r^2}) g(4r)\, dr \\ &\leq 4^{N-2} g(t) \prod_{i=2}^{k} (1 + C\alpha_i(t)) + 4^{N-2} C g(t) \prod_{i=2}^{k} (1 + C\alpha_i(t)) \int_{t/4^{k+1}}^{t/4^k} (1 + \Lambda_0 + \chi_{\{2r \geq d_0\}} \frac{d_0}{4r^2})\, dr \\ &= 4^{N-2} g(t) \prod_{i=2}^{k+1} (1 + C\alpha_i(t)). \end{aligned}$$

The conclusion then follows from the definition of $\alpha_i(t)$ and the inequality, valid for all $m \in \mathbb{N}$,

$$\begin{aligned} \prod_{i=2}^{m} (1 + C\alpha_i(t)) \leq \exp\Big(C \sum_{i=2}^{\infty} \alpha_i(t)\Big) &= \exp\Big(C \int_0^{t/4} (1 + \Lambda_0 + \chi_{\{2r \geq d_0\}} \frac{d_0}{4r^2})\, dr\Big) \\ &= \exp\Big(C((1 + \Lambda_0)\frac{t}{4} + 2d_0 \chi_{\{t \geq 2d_0\}}(\frac{1}{2d_0} - \frac{1}{t}))\Big). \end{aligned}$$

Coming back to h, we deduce

$$h(s) \leq g(s) \leq 4^{N-2} \exp\Big(C((1+\Lambda_0)\frac{t}{4} + 2d_0\chi_{\{t\geq 2d_0\}}(\frac{1}{2d_0} - \frac{1}{t}))\Big)(h(t)+D),$$

and the proof is complete. $\square$

To conclude the proof of Lemma 3.13, we apply Lemma 3.14 with $s_1 = r_\varepsilon$, $s_2 = r$, $h = \bar{E}_\varepsilon$, $C = C$ and $D = \Lambda_0\varepsilon^\beta$. The first hypothesis needed is still verified for the modified scaled energy $\bar{E}_\varepsilon$ and the second one is (77). We then infer that, for every $r_\varepsilon \leq s < r$,

$$\bar{E}_\varepsilon(s) \leq C \exp\Big(C((1+\Lambda_0)\frac{t}{4} + 2d_0\chi_{\{t\geq 2d_0\}}(\frac{1}{2d_0} - \frac{1}{t}))\Big)(\bar{E}_\varepsilon(r) + \Lambda_0\varepsilon^\beta).$$

This finishes the proof of Lemma 3.13. $\square$

3.3 Proofs of Propositions 1 and 2

Before giving the proof, we notice that for any $d_0 \geq 0$ and $t \geq 0$,

$$0 \leq d_0\chi_{\{t\geq d_0\}}(\frac{1}{d_0} - \frac{1}{t}) \leq 1,$$

since, for $t \geq d_0 > 0$, $d_0(1/d_0 - 1/t) = (1 - d_0/t) \in [0,1]$. Therefore, in the Neumann case, this extra term is less than a constant. We assume $x_0 = 0$ and first consider the case

$$\theta r \leq \rho := (1 + \Lambda_0|\log\varepsilon|)^{-1} \leq r/2. \tag{81}$$

By Lemma 3.4 (resp. Lemma 3.11) in the Dirichlet case (resp. the Neumann case), we deduce

$$\tilde{E}_\varepsilon(\theta r) \leq C(\tilde{E}_\varepsilon(\rho) + T^\nu_\varepsilon(\theta r, \rho) + \Lambda_0^2\varepsilon^2|\log\varepsilon|^4) \tag{82}$$

$$(\text{resp. } \tilde{E}_\varepsilon(\theta r) \leq C(\tilde{E}_\varepsilon(\rho) + \Lambda_0^2\varepsilon^2|\log\varepsilon|^4)). \tag{83}$$

Next, by Lemma 3.7 (resp. Lemma 3.13), recalling $r_\varepsilon \leq \rho$ for $0 < \varepsilon < \varepsilon_0$ sufficiently small, applied with $s = \rho$ and $r/2$,

$$\tilde{E}_\varepsilon(\rho) \leq \bar{E}_\varepsilon(\rho) \leq C(\tilde{E}_\varepsilon(r) + T^\nu_\varepsilon(\rho, r/2) + \Lambda_0\varepsilon^\beta) \tag{84}$$

$$(\text{resp. } \tilde{E}_\varepsilon(\rho) \leq \bar{E}_\varepsilon(\rho) \leq C(\tilde{E}_\varepsilon(r) + \Lambda_0\varepsilon^\beta)). \tag{85}$$

Combining (82) and (84) (resp. (83) and (85)) yields (17) (resp. (18)) if (81) holds. If

$$\theta r \leq r/2 \leq \rho,$$

we only use Lemma 3.4 (resp. Lemma 3.11) as for (82) (resp. (83)), and if

$$\rho \leq \theta r \leq r/2,$$

we only use Lemma 3.7 (resp. Lemma 3.13) as for (84) (resp. (85)). The proof is complete. $\square$

4 Proof of Theorems 2 and 3

We follow step by step the lines of [BBO] (Theorem 2 bis) and [BOS] (Theorem 2). The proof is divided in three parts. Let $0 < \delta < 1/32$ be a constant to be determined later, depending only on N and Ω, and, in the Dirichet case, on ν and the constant C in (12).

4.1 Proof of Theorem 2

Part A: Choosing a "good" radius.

Lemma 4.1. *Assume $0 < \varepsilon < \delta^{1/(2\beta)}$, that u is a solution of (1)-(5) and that*

$$\tilde{E}_\varepsilon(\bar{r}) \leq \eta|\log\varepsilon| \qquad \text{and} \qquad T_\varepsilon^\nu(x_0, r_\varepsilon, \bar{r}) \leq \eta|\log\varepsilon| \tag{86}$$

holds for a $r_\varepsilon^{1/2} \leq \bar{r} \leq \min(R, (1+\Lambda_0)^{-1/\nu})$. Then, there exists a radius $r_0 \in (r_\varepsilon, r_\varepsilon^{1/2})$ such that

- $\displaystyle\frac{1}{r_0^{N-2}}\int_{\check{B}_{r_0}} \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2} \leq C(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|,$
- $\tilde{E}_\varepsilon(r_0) - 2^{N-2}\tilde{E}_\varepsilon(\delta r_0) \leq C(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|.$

Proof. From (53), we have for $r_\varepsilon \leq r \leq \min(R, (1+\Lambda_0)^{-1})$

$$\frac{d\bar{E}_\varepsilon}{dr} \geq A(r) - C\Big((1+\Lambda_0)\frac{\bar{E}_\varepsilon(4r)}{(4r)^{1-\nu}} + \frac{G_\varepsilon(2r)}{r^\nu} + \Lambda_0\varepsilon^\beta\Big), \tag{87}$$

where $A(r)$ is defined in (54). Let k be the greatest integer such that $r_\varepsilon(\frac{\delta}{4})^{-k} \leq \bar{r}/8$ and define the intervals

$$I_j := \Big(r_\varepsilon(\frac{\delta}{4})^{-j+1}, r_\varepsilon(\frac{\delta}{4})^{-j}\Big), \qquad 1 \leq j \leq k.$$

These intervals are clearly disjoint and $\cup_{j=1}^k I_j \subset (r_\varepsilon, \bar{r}/8)$. From $\bar{r} \geq r_\varepsilon^{1/2}$ and

$$|\log r_\varepsilon| \geq C^{-1}|\log\varepsilon|,$$

we infer

$$k \geq C^{-1}\frac{|\log\varepsilon|}{|\log\delta|}. \tag{88}$$

We integrate (87) over each I_j, $1 \leq j \leq k$, and use the monotonicity formula of Proposition 1

$$\begin{aligned}\sum_{j=1}^k \int_{I_j} A(r) - &C\Big((1+\Lambda_0)\frac{\bar{E}_\varepsilon(4r)}{(4r)^{1-\nu}} + \frac{G_\varepsilon(2r)}{r^\nu} + \Lambda_0\varepsilon^\beta\Big)\, dr \\ &\leq \sum_{j=1}^k \bar{E}_\varepsilon(r_\varepsilon(\frac{\delta}{4})^{-j+1}) - \bar{E}_\varepsilon(r_\varepsilon(\frac{\delta}{4})^{-j}) \\ &\leq C(\tilde{E}_\varepsilon(\bar{r}) + T_\varepsilon^\nu(r_\varepsilon, \bar{r}) + \Lambda_0\varepsilon^\beta) \\ &\leq C(\eta|\log\varepsilon| + \Lambda_0\varepsilon^\beta)\end{aligned} \tag{89}$$

by hypothesis. Moreover, still with the monotonicity formula of Proposition 1, we have

$$\sum_{j=1}^{k}\int_{I_j}\Big((1+\Lambda_0)\frac{\bar{E}_\varepsilon(4r)}{(4r)^{1-\nu}}+\frac{G_\varepsilon(2r)}{r^\nu}+\Lambda_0\varepsilon^\beta\Big)\,dr$$

$$\leq C(1+\Lambda_0)(\tilde{E}_\varepsilon(\bar{r})+T^\nu_\varepsilon(r_\varepsilon,\bar{r})+\Lambda_0\varepsilon^\beta)\int_{r_\varepsilon}^{\bar{r}}\frac{dr}{r^{1-\nu}}+CT^\nu_\varepsilon(r_\varepsilon,\bar{r})+C\Lambda_0\varepsilon^\beta$$
$$\leq C(\eta|\log\varepsilon|+\Lambda_0\varepsilon^\beta), \tag{90}$$

by (86) and the hypothesis $\bar{r}^\nu\leq(1+\Lambda_0)^{-1}$. We deduce from (88), (89) and (90) the existence of some $j_0\in\{1,\dots,k\}$ such that

$$\int_{I_{j_0}}A(r)\leq\bar{E}_\varepsilon(r_\varepsilon(\frac{\delta}{4})^{-j_0+1})-\bar{E}_\varepsilon(r_\varepsilon(\frac{\delta}{4})^{-j_0})\leq C(\eta+\Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|. \tag{91}$$

In particular, by the mean value formula, there exists some

$$r_0\in\Big(\frac{r_\varepsilon}{2}(\frac{\delta}{4})^{-j_0},r_\varepsilon(\frac{\delta}{4})^{-j_0}\Big)\subset I_{j_0}$$

such that

$$\frac{1}{r_0^{N-2}}\int_{\check{B}_{r_0}}\frac{(a_\varepsilon-|u|^2)^2}{2\varepsilon^2}\leq C(\eta+\Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|,$$

which is the first assertion of the Lemma. Noticing that $\frac{\delta}{2}r_0\in I_{j_0}$, we infer from (91)

$$\tilde{E}_\varepsilon(r_0)-2^{N-2}\tilde{E}_\varepsilon(\delta r_0)\leq\bar{E}_\varepsilon(r_0)-\bar{E}_\varepsilon(\frac{\delta}{2}r_0)\leq C(\eta+\Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|,$$

where we have used once more Lemma 3.7. This is the second assertion of the Lemma. $\square$

Part B: δ-energy decay.

Lemma 4.2. *There exist constants C and ε_0, depending on N, ν and Ω, such that, if u is a solution of* (1)-(5) *with g_ε satisfying* (12) *and*

$$T^\nu_\varepsilon(x_0,\varepsilon,r_\varepsilon^{1/2})\leq\eta,$$

with $\varepsilon<\varepsilon_0$ and $\varepsilon<r\leq r_\varepsilon^{1/2}$, then

$$E_\varepsilon(\delta r)\leq C\Big(\gamma^2+\delta^N+\frac{\gamma^{-4}}{r^{N-2}}\int_{\check{B}_r}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\Big)E_\varepsilon(r)+C\gamma^{-2}\int_{\check{B}_r}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}$$
$$+Cr^{N-2}\Big(1+\frac{\gamma^{-4}}{r^{N-2}}\int_{\check{B}_r}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\Big)(\eta+\Lambda_0\varepsilon^\beta)+C\sqrt{\eta}r^{N-2}.$$

For the ease of presentation, we will assume that Ω is locally the half plane $\mathbb{R}^N_+=\mathbb{R}^{N-1}\times\mathbb{R}_+$. By the mean-value inequality, there exists $r/32\leq r_1\leq r/16$ such that

$$r\int_{\Omega\cap\partial B_{r_1}(x_0)}|\nabla u|^2\leq 96\int_{\check{B}_r(x_0)}|\nabla u|^2, \tag{92}$$
$$r\int_{\Omega\cap\partial B_{r_1}(x_0)}(a_\varepsilon-|u|^2)^2\leq 96\int_{\check{B}_r(x_0)}(a_\varepsilon-|u|^2)^2, \tag{93}$$
$$r_1^{N-1}G_\varepsilon(r_1)\leq 96\eta r^{N-2}. \tag{94}$$

To clarify the last one, if $\rho^{N-1}G_\varepsilon(\rho) \geq 96\eta r^{N-2}$, then $\rho^{-\nu}G_\varepsilon(\rho) \geq 96\eta\rho^{-1-\nu} \geq 96\eta\varepsilon^{-1}$. The proof is divided in four steps.

Step 1: Hodge-de Rham decomposition of $u \times \nabla u$.

Since u is a solution of (1) and div $\vec{c} = 0$,

$$d^*(u \times du) = u \times (-\Delta u) = -(u, \vec{c} \cdot \nabla u)|\log \varepsilon| = d^*\Big(|\log \varepsilon|\frac{|u|^2 - 1}{2}c\Big), \tag{95}$$

where $c := \sum_{i=1}^N c_i(x)dx_i$. We consider the solution of the auxiliary problem

$$\begin{cases} \Delta\xi = 0 & \text{in } \check{B}_{r_1}(x_0), \\ \frac{\partial \xi}{\partial n} = u \times \frac{\partial u}{\partial n} - |\log \varepsilon|\frac{|u|^2-1}{2}\vec{c}\cdot n & \text{on } \mathbb{R}^N_+ \cap \partial B_{r_1}(x_0), \\ \xi = 0 & \text{on } \partial\mathbb{R}^N_+ \cap B_{r_1}(x_0), \end{cases}$$

which exists and is unique. By (92) and (93), we have

$$\begin{aligned} \int_{\check{B}_{r_1}} |\nabla\xi|^2 &\leq Cr\Big(\int_{\partial\mathbb{R}^N_+\cap B_{r_1}(x_0)} \Big|\frac{\partial u}{\partial n}\Big|^2 + \Lambda_0^2\varepsilon^2|\log\varepsilon|^2 \int_{\partial\mathbb{R}^N_+\cap B_{r_1}(x_0)} \frac{(1-|u|^2)^2}{\varepsilon^2}\Big) \\ &\leq C(E_\varepsilon(r) + r^N\Lambda_0\varepsilon^\beta), \end{aligned} \tag{96}$$

from which we infer by standard estimates

$$\int_{\check{B}_{\delta r}} |\nabla\xi|^2 \leq C\delta^N \int_{\check{B}_{r_1}} |\nabla\xi|^2 \leq C\delta^N(E_\varepsilon(r) + r^N\Lambda_0\varepsilon^\beta). \tag{97}$$

We turn now to the Hodge-de Rham decomposition of $u \times du$. By construction of ξ and from (95), we have in $\mathcal{D}'(\mathbb{R}^N_+)$

$$d^*\Big[\Big(u \times du - |\log\varepsilon|\frac{|u|^2-1}{2}c - d\xi\Big)\chi\Big] = 0. \tag{98}$$

By classical Hodge theory (see for instance the Appendix of [BBO], Proposition A.8), there exists some 2-form φ on $\mathbb{R}^N_+$ such that

$$d^*\varphi = \Big(u \times du - |\log\varepsilon|\frac{|u|^2-1}{2}c - d\xi\Big)\chi \quad \text{in } \mathcal{D}'(\mathbb{R}^N_+), \tag{99}$$

$$d\varphi = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N_+), \tag{100}$$

$$||\nabla\varphi||^2_{L^2(\mathbb{R}^N_+)} \leq C(E_\varepsilon(r_1) + ||\nabla\xi||^2_{L^2(\check{B}_{r_1})} + r^N\Lambda_0\varepsilon^\beta), \tag{101}$$

$$\varphi_\top = 0 \quad \text{on } \partial\mathbb{R}^N_+, \tag{102}$$

$$|\varphi(x)| \cdot |x|^{N-1} \to 0 \quad \text{as } |x| \to +\infty. \tag{103}$$

Step 2: Improved estimates for $\nabla\varphi$ on $\check{B}_{\delta r}(x_0)$.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be any smooth function such that

$$\begin{cases} f(t) = \frac{1}{t} & \text{if } 1-\gamma \leq t \leq 1+\gamma, \\ f(t) = 1 & \text{if } t \leq 1-2\gamma \text{ or } t \geq 1+2\gamma, \\ |f'(t)| \leq 4 & \text{for any } t \geq 0. \end{cases}$$

We consider the function on $\mathbb{R}^N_+$ defined by

$$\tau(x) := \begin{cases} f^2(|u(x)|) & \text{in } \check{B}_{r_1}, \\ 1 & \text{outside,} \end{cases}$$

so that, by construction,

$$0 \le \tau - 1 \le 4\gamma \quad \text{ in } \mathbb{R}^N_+. \tag{104}$$

Note that

$$f^2(|u|)u \times du = f(|u|)u \times d(f(|u|)u),$$

thus, in $\check{B}_{r_1}$,

$$d(\tau u \times du) = d[f(|u|)u \times d(f(|u|)u)] = 2\sum_{i<j} \partial_i(f(|u|)u) \times \partial_j(f(|u|)u)\ dx_i \wedge dx_j.$$

Turning now to φ, we apply the d operator to (99) to deduce that, in $\mathcal{D}'(\mathbb{R}^N_+)$,

$$\begin{aligned} -\Delta\varphi = dd^*\varphi &= d(\chi\tau u \times du) - d(\chi d\xi) - d\Big(\chi|\log\varepsilon|\frac{|u|^2-1}{2}c\Big) + d(\chi(1-\tau)u \times du) \\ &= \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5, \end{aligned}$$

where, χ standing for the characteristic function of $\check{B}_{r_1}(x_0)$,

$$\begin{aligned} \omega_1 &:= \chi d(\tau u \times du) = 2\chi\sum_{i<j} \partial_i(f(|u|)u) \times \partial_j(f(|u|)u)\ dx_i \wedge dx_j, \\ \omega_2 &:= \sigma_{\partial B_{r_1}\cap\mathbb{R}^N_+} dr \wedge (f^2(|u|)u \times du) - \sigma_{B_{r_1}\cap\partial\mathbb{R}^N_+} dx_N \wedge (f^2(|g_\varepsilon|)g_\varepsilon \times dg_\varepsilon) \qquad (r = |x - x_0|), \\ \omega_3 &:= -d(\chi d\xi) = -\sigma_{\partial B_{r_1}\cap\mathbb{R}^N_+} dr \wedge d\xi, \\ \omega_4 &:= -d\Big(\chi|\log\varepsilon|\frac{|u|^2-1}{2}c\Big), \\ \omega_5 &:= d(\chi(1-\tau)u \times du) \end{aligned}$$

and σ stands for surface measure. We denote also the 1-forms on $\partial\mathbb{R}^N_+$

$$A_1 = A_2 = A_3 = 0, \qquad A_4 := -\chi_\top|\log\varepsilon|\frac{|g_\varepsilon|^2-1}{2}c_\top \qquad \text{and} \qquad A_5 := \chi_\top(1 - f^2(|g_\varepsilon|))g_\varepsilon \times dg_\varepsilon.$$

From the Appendix of [BBO], we know that the solutions of the problems on $\mathbb{R}^N_+$

$$\begin{cases} \Delta\varphi_i &= \omega_i & \text{in } \mathbb{R}^N_+, \\ (\varphi_i)_\top &= 0 & \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{0\}, \\ (d^*\varphi_i)_\top &= A_i & \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{0\}, \end{cases}$$

for $i = 1, 2, 3, 4$ and 5 exist in $H^1_{loc}(\mathbb{R}^N_+)$, but are not unique for $i = 4$ and 5. We will consider in this case the solutions given by convolutions (the ω_i's and A_i's have compact support). Note that this prevents us from imposing condition at infinity since, *a priori*, the integrals over $\partial\mathbb{R}^N_+$ of the components of A_4 and A_5 are not zero. Concerning φ_2, we note that the second measure in ω_2

involves a measure supported on $\partial\mathbb{R}^N_+$, but the weak formulation of the equation has a meaning for test functions in $\mathcal{C}^\infty_{\top,c}(\Lambda^2\bar{\mathbb{R}}^N_+)$ (see Lemma A.4 in [BBO]), namely, for all $\zeta \in \mathcal{C}^\infty_{\top,c}(\Lambda^2\bar{\mathbb{R}}^N_+)$,

$$\langle d\varphi_2, d\zeta\rangle + \langle d^*\varphi_2, d^*\zeta\rangle = \langle \omega_2, \zeta\rangle = \int_{\mathbb{R}^N_+\cap\partial B_{r_1}} dr\wedge f^2(|u|)u\times du\wedge(\star\zeta)_\top - \int_{\partial\mathbb{R}^N_+} (f^2(|g_\varepsilon|)g_\varepsilon\times dg_\varepsilon)\wedge(\star\zeta)_\top.$$

Let $\phi := \varphi - \sum_{i=1}^5 \varphi_i$. Then, by (99) and the condition $\xi = 0$ on $B_{r_1}(x_0)\cap\mathbb{R}^N_+$,

$$(d^*\phi)_\top = (d^*\varphi)_\top - A_4 - A_5 = \chi_\top f^2(|g_\varepsilon|)g_\varepsilon \times dg_\varepsilon =: A.$$

Consequently, ϕ is the solution given by convolution of

$$\begin{cases} \Delta\phi & = 0 \qquad \text{in } \mathbb{R}^N_+, \\ \phi_\top & = 0 \qquad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times\{0\}, \\ (d^*\phi)_\top & = A \qquad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times\{0\}, \end{cases} \tag{105}$$

and

$$\varphi = \phi + \sum_{i=1}^5 \varphi_i.$$

We turn now to estimate ϕ and the φ_i's.

Estimate for ϕ. We have

$$\int_{\breve{B}_{r_1}(x_0)} |\nabla\phi|^2 \le Cr\int_{B_1 r(x_0)\cap\partial\mathbb{R}^N_+} |\nabla_\top g_\varepsilon|^2 \le Cr_1^{N-1}G_\varepsilon(r_1) \le C\eta r^{N-2}. \tag{106}$$

This is a direct consequence of standard estimates (see the Appendix of [BBO]) and a scaling argument for the equation (105) combined with the bound

$$\|A\|^2_{L^2(\partial\mathbb{R}^N_+)} \le C\int_{B_{r_1}(x_0)\cap\partial\mathbb{R}^N_+} |\nabla_\top g_\varepsilon|^2.$$

Estimate for φ_5. We claim that

$$\int_{\mathbb{R}^N_+} |\nabla\varphi_5|^2 \le C\gamma^2\int_{\breve{B}_r} |\nabla u|^2 \le C\gamma^2 E_\varepsilon(r). \tag{107}$$

Indeed, since φ_5 is a solution of

$$-\Delta\varphi_5 = \omega_5 = d(\chi(1-\tau)u\times du),$$

we obtain, multiplying by φ_5 and integrating (see Lemma A.4 in [BBO])

$$\|\nabla\varphi_5\|^2_{L^2(\mathbb{R}^N_+)} = \int_{\mathbb{R}^N_+} \langle d(\chi(1-\tau)u\times du), \varphi_5\rangle - \int_{\partial\mathbb{R}^N_+} A_5\wedge(\star\varphi_5)_\top.$$

Moreover, integration by parts once more yields

$$\int_{\mathbb{R}^N_+} \langle d(\chi(1-\tau)u\times du), \varphi_5\rangle = \int_{\mathbb{R}^N_+} \langle \chi(1-\tau)u\times du, d^*\varphi_5\rangle + \int_{\partial\mathbb{R}^N_+} (\chi_\top(1-f^2(|g_\varepsilon|))g_\varepsilon\times dg_\varepsilon)\wedge(\star\varphi_5)_\top.$$

Thus, by definition of A_5 (this was done for that purpose !),

$$\|\nabla\varphi_5\|_{L^2(\mathbb{R}^N_+)} = \int_{\mathbb{R}^N_+} \langle \chi(1-\tau)u\times du, d^*\varphi_5\rangle \le C\|1-\tau\|_{L^\infty(\check{B}_{r_1})}\|u\|_{L^\infty(\check{B}_{r_1})}\|\nabla u\|_{L^2(\check{B}_{r_1})}$$

and the result comes from Lemma 1 and (104).

Estimate for φ_4. We have

$$\int_{\mathbb{R}^N_+} |\nabla\varphi_4|^2 \le C\Big(\int_{\check{B}_{r_1}} \frac{(a_\varepsilon-|u|^2)^2}{2\varepsilon^2} + r^N\Lambda_0\varepsilon^\beta\Big). \tag{108}$$

Since φ_4 satisfies the equation

$$-\Delta\varphi_4 = \omega_4 = -d\Big(\chi|\log\varepsilon|\frac{|u|^2-1}{2}c\Big),$$

we argue as for φ_5, that is multiplying by φ_4 and using the definition of A_4, to obtain

$$\int_{\mathbb{R}^N_+} |\nabla\varphi_4|^2 \le C\Lambda_0^2\varepsilon^2|\log\varepsilon|^2\int_{\check{B}_{r_1}} \frac{(1-|u|^2)^2}{2\varepsilon^2} \le C\Big(\int_{\check{B}_{r_1}} \frac{(a_\varepsilon-|u|^2)^2}{2\varepsilon^2} + r^N\Lambda_0\varepsilon^\beta\Big),$$

which is the claim.

Estimate for φ_3. We claim that

$$\int_{\check{B}_{\delta r}} |\nabla\varphi_3|^2 \le C\delta^N(E_\varepsilon(r) + r^N\Lambda_0\varepsilon^\beta). \tag{109}$$

Indeed, we have first by (97) and arguing as for φ_5 since $\xi_\top = 0$ on $\partial\mathbb{R}^N_+\cap B_{r_1}(x_0)$,

$$\int_{\mathbb{R}^N_+} |\nabla\varphi_3|^2 \le C\int_{\partial B_{r_1}\cap\mathbb{R}^N_+} |\nabla\xi|^2 \le C(E_\varepsilon(r) + r^N\Lambda_0\varepsilon^\beta).$$

Next, we note that ω_3 has support in $\partial B_{r_1}\cap\mathbb{R}^N_+$, thus φ_3 is harmonic inside $\check{B}_{r_1}$ and thus by standard estimates (and scaling),

$$\|\nabla\varphi_3\|_{L^\infty(\check{B}_{r/32})} \le Cr^{1-N/2}\|\nabla\varphi_3\|_{L^2(\check{B}_{r_1})},$$

from which ($\delta\le 1/32$) we infer (109).

Estimate for φ_2. We have

$$\int_{\check{B}_{\delta r}} |\nabla\varphi_2|^2 \le C\delta^N E_\varepsilon(r) + Cr_1^{N-1}G_\varepsilon(r_1) \le C\delta^N E_\varepsilon(r) + C\eta r^{N-2}. \tag{110}$$

We write

$$\omega_2 = \sigma_{\partial B_{r_1}\cap\mathbb{R}^N_+} dr\wedge(f^2(|u|)u\times du) - \sigma_{B_{r_1}\cap\partial\mathbb{R}^N_+} dx_N\wedge(f^2(|g_\varepsilon|)g_\varepsilon\times dg_\varepsilon) = \omega_{2,1}+\omega_{2,2}$$

and thus write with obvious notations $\varphi_2 = \varphi_{2,1}+\varphi_{2,2}$. The estimate for $\varphi_{2,1}$

$$\int_{\check{B}_{\delta r}} |\nabla\varphi_{2,1}|^2 \le C\delta^N E_\varepsilon(r)$$

follows as for φ_3. Concerning $\varphi_{2,2}$, we have

$$\int_{\mathbb{R}^N_+} |\nabla \varphi_{2,2}|^2 \le C r_1^{N-1} G_\varepsilon(r_1),$$

and the conclusion follows from these two inequalities and (94).

Estimate for φ_1. The crucial estimate is

$$|\omega_1| \le \frac{C}{\gamma^2} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \quad \text{ in } \check{B}_r. \tag{111}$$

Indeed,

$$\omega_1 = 2\chi \sum_{i<j} \partial_i(f(|u|)u) \times \partial_j(f(|u|)u)\ dx_i \wedge dx_j.$$

If $1-\gamma \le |u| \le 1+\gamma$, since $f(|u|) = 1/|u|$, two partial derivatives $\partial_i(f(|u|)u)$ and $\partial_j(f(|u|)u)$ are both tangent to $\mathbb{S}^1$ at $\frac{u}{|u|}$ thus are colinear, and therefore $\omega_1 = 0$.
If $|u| < 1-\gamma$ or $|u| > 1+\gamma$, by Lemma 1,

$$|\omega_1| \le \frac{C}{\varepsilon^2} \le \frac{C}{\varepsilon^2\gamma^2}(1-|u|^2)^2$$

and the conclusion follows from

$$(1-|u|^2)^2 \le 2(a_\varepsilon - |u|^2)^2,$$

valid at least if $0 < \varepsilon < \varepsilon_0$ sufficiently small (depending on γ and Λ_0).
Next, we claim that

$$||\varphi_1||_{L^\infty(\mathbb{R}^N_+)} \le \frac{C}{\gamma^2 r^{N-2}}\Big(E_\varepsilon(r) + r^{N-2}T^\nu_\varepsilon(\varepsilon, r_\varepsilon^{1/2}) + \Lambda_0 r^{N-2}\varepsilon^\beta\Big). \tag{112}$$

Indeed, we know that (cf. Proposition A.3 in [BBO])

$$|\varphi_1(x)| \le 2c_N \int_{\mathbb{R}^N_+} \frac{|\omega_1(y)|}{|x-y|^{N-2}} dy,$$

thus, using (111),

$$|\varphi_1(x)| \le \frac{C}{\gamma^2} \int_{\check{B}_{r_1}} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2 |x-y|^{N-2}} dy. \tag{113}$$

Since φ_1 is harmonic outside $\check{B}_{r_1}$ and tends to 0 at infinity, we deduce by the maximum principle

$$\|\varphi_1\|_{L^\infty(\mathbb{R}^N_+)} = \|\varphi_1\|_{L^\infty(\check{B}_{r_1})}.$$

In order to prove (112), it suffices then to prove

$$\|\varphi_1\|_{L^\infty(\check{B}_{r_1})} \le \frac{C}{\gamma^2 r^{N-2}}\Big(E_\varepsilon(r) + r^{N-2}T^\nu_\varepsilon(\varepsilon, r_\varepsilon^{1/2}) + \Lambda_0 r^{N-2}\varepsilon^\beta\Big).$$

Let $x \in \check{B}_{r_1}$. Since $\check{B}_{r_1}(x_0) \subset \check{B}_{r/4}(x)$, we deduce from (113)

$$
\begin{aligned}
|\varphi_1(x)| &\leq \frac{C}{\gamma^2} \int_{\check{B}_{r/4}(x)} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2 |x-y|^{N-2}} dy = \frac{C}{\gamma^2} \int_0^{r/4} \frac{1}{\rho^{N-2}} \int_{\Omega \cap \partial B_\rho(x)} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \, d\rho \\
&= \frac{C}{\gamma^2}(N-2) \int_0^{r/4} \frac{1}{\rho^{N-1}} \int_{\Omega \cap B_\rho(x)} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \, d\rho + \frac{C}{\gamma^2} \Big[\frac{1}{\rho^{N-2}} \int_{\Omega \cap B_\rho(x)} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \Big]_0^{r/4}.
\end{aligned}
$$

Using the monotonicity formula of Lemma 3.4 (for $0 \leq \rho \leq r/4 \leq (1 + \Lambda_0 |\log \varepsilon|)^{-1}$), we obtain

$$
|\varphi_1(x)| \leq \frac{C}{\gamma^2} \Big(\tilde{E}_\varepsilon(x, \frac{r}{4}) + T^\nu_\varepsilon(\varepsilon, r) + \Lambda_0 \varepsilon^\beta \Big) \leq \frac{C}{\gamma^2 r^{N-2}} \Big(E_\varepsilon(r) + r^{N-2}\eta + \Lambda_0 r^{N-2} \varepsilon^\beta \Big)
$$

since $\check{B}_{r/4}(x) \subset \check{B}_r(x_0)$, and the proof of (112) is complete.

To conclude, we go back to the equation

$$
-\Delta \varphi_1 = \omega_1
$$

to deduce

$$
\int_{\mathbb{R}^N_+} |\nabla \varphi_1|^2 \leq \|\varphi_1\|_{L^\infty(\mathbb{R}^N_+)} \int_{\check{B}_{r_1}} |\omega_1| \leq \|\varphi_1\|_{L^\infty(\mathbb{R}^N_+)} \int_{\check{B}_r} |\omega_1|,
$$

since $r_1 \leq r$, so that, by (111) and (112),

$$
\int_{\mathbb{R}^N_+} |\nabla \varphi_1|^2 \leq \frac{C}{\gamma^4} \Big(\frac{1}{r^{N-2}} \int_{\check{B}_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \Big) \Big(E_\varepsilon(r) + r^{N-2}\eta + r^{N-2} \Lambda_0 \varepsilon^\beta \Big). \tag{114}
$$

Step 2 completed. Combining the estimates for ϕ and φ_i, $1 \leq i \leq 5$, we are led to, for $0 < \delta < 1/32$,

$$
\begin{aligned}
\int_{\check{B}_{\delta r}(x_0)} |\nabla \varphi|^2 \leq C \Big(\gamma^2 + \delta^N + \frac{\gamma^{-4}}{r^{N-2}} \int_{\check{B}_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \Big) E_\varepsilon(r) + C \int_{\check{B}_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \\
+ C r^{N-2} \Big(1 + \frac{\gamma^{-4}}{r^{N-2}} \int_{\check{B}_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \Big) (\eta + \Lambda_0 \varepsilon^\beta).
\end{aligned} \tag{115}
$$

Step 3: Improved estimates for $\nabla(|u|^2)$ on $\check{B}_{\delta r}(x_0)$.

The equation for $|u|^2$ reads

$$
\Delta |u|^2 + 2 \frac{(a_\varepsilon - |u|^2)|u|^2}{\varepsilon^2} = 2|\nabla u|^2 + 2|\log \varepsilon| (i\vec{c} \cdot \nabla u, u).
$$

Multiplying by $a_\varepsilon - |u|^2$ and integrating over $\check{B}_{r_1}(x_0)$, we obtain

$$
\begin{aligned}
\int_{\check{B}_{r_1}} \big|\nabla |u|^2\big|^2 + 2 \frac{(a_\varepsilon - |u|^2)^2 |u|^2}{\varepsilon^2} = 2 \int_{\check{B}_{r_1}} (a_\varepsilon - |u|^2) |\nabla u|^2 + \int_{\partial \check{B}_{r_1}} (a_\varepsilon - |u|^2) \frac{\partial |u|^2}{\partial n} \\
+ \int_{\check{B}_{r_1}} \nabla |u|^2 \cdot \nabla a_\varepsilon + \int_{\check{B}_{r_1}} 2|\log \varepsilon| (i\vec{c} \cdot \nabla u, u)(a_\varepsilon - |u|^2).
\end{aligned} \tag{116}
$$

For the second term in the right-hand side of (116), we have first by (92) and (93)

$$\begin{aligned}\Big|\int_{\mathbb{R}^N_+\cap\partial B_{r_1}}(a_\varepsilon-|u|^2)\frac{\partial|u|^2}{\partial n}\Big| &\le C\varepsilon\Big(\int_{\mathbb{R}^N_+\cap\partial B_{r_1}}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\Big)^{1/2}\Big(\int_{\mathbb{R}^N_+\cap\partial B_{r_1}}|\nabla u|^2\Big)^{1/2}\\ &\le C\frac{\varepsilon}{r}\Big(\int_{\mathbb{R}^N_+\cap B_{r_1}}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\Big)^{1/2}\Big(\int_{\mathbb{R}^N_+\cap B_{r_1}}|\nabla u|^2\Big)^{1/2}\\ &\le C\gamma^2\int_{\check{B}_{r_1}}|\nabla u|^2+C\gamma^{-2}\int_{\check{B}_{r_1}}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\end{aligned}$$

since $r\ge\varepsilon$, and next, using Lemma 1, Cauchy-Schwarz and (94), we have, since $\varepsilon\le r\le r_\varepsilon^{1/2}$,

$$\begin{aligned}\Big|\int_{\partial\mathbb{R}^N_+\cap B_{r_1}}(a_\varepsilon-|u|^2)\frac{\partial|u|^2}{\partial n}\Big| &\le C\int_{\partial\mathbb{R}^N_+\cap B_{r_1}}\frac{\big|a_\varepsilon-|u|^2\big|}{\varepsilon}\\ &\le C\Big(r_1^{N-1}\int_{\partial\mathbb{R}^N_+\cap B_{r_1}}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\Big)^{1/2}\\ &\le C\Big(r_1^{N-1}r_1^{N-2}G_\varepsilon(r_1)\Big)^{1/2}\\ &\le C\sqrt{\eta}r^{N-2}.\end{aligned}$$

As a consequence,

$$\Big|\int_{\partial\check{B}_{r_1}}(a_\varepsilon-|u|^2)\frac{\partial|u|^2}{\partial n}\Big|\le C\gamma^2\int_{\check{B}_r}|\nabla u|^2+C\gamma^{-2}\int_{\check{B}_r}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}+C\sqrt{\eta}r^{N-2}. \tag{117}$$

We also have

$$\begin{aligned}\Big|\int_{\check{B}_{r_1}}(a_\varepsilon-|u|^2)|\nabla u|^2\Big| &\le \int_{\check{B}_{r_1}\cap\{|a_\varepsilon-|u|^2|\le\gamma^2\}}\big|a_\varepsilon-|u|^2\big|\cdot|\nabla u|^2\\ &\quad+\int_{\check{B}_{r_1}\cap\{|a_\varepsilon-|u|^2|>\gamma^2\}}\big|a_\varepsilon-|u|^2\big|\cdot|\nabla u|^2\\ &\le \gamma^2\int_{\check{B}_r}|\nabla u|^2+\frac{C}{\gamma^2}\int_{\check{B}_r}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2},\end{aligned} \tag{118}$$

where we have used Lemma 1 for the second term. Moreover, by Lemma 1 and (8),

$$\begin{aligned}\Big|\int_{\check{B}_{r_1}}2|\log\varepsilon|(i\vec{c}\cdot\nabla u,u)(a_\varepsilon-|u|^2)\Big| &\le C\Lambda_0\varepsilon|\log\varepsilon|\Big(\int_{\check{B}_r}|\nabla u|^2\Big)^{1/2}\Big(\int_{\check{B}_r}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\Big)^{1/2}\\ &\le C\gamma^2\int_{\check{B}_r}|\nabla u|^2+\frac{C}{\gamma^2}\int_{\check{B}_r}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}.\end{aligned} \tag{119}$$

Finally, using (8),

$$\begin{aligned}\Big|\int_{\check{B}_{r_1}}\nabla|u|^2\cdot\nabla a_\varepsilon\Big| &\le \frac{1}{2}\int_{\check{B}_r}\big|\nabla(|u|^2)\big|^2+\frac{\varepsilon^4|\log\varepsilon|^4}{2}\int_{\check{B}_r}|\nabla d|^2\\ &\le \frac{1}{2}\int_{\check{B}_r}\big|\nabla(|u|^2)\big|^2+Cr^N\Lambda_0\varepsilon^\beta.\end{aligned} \tag{120}$$

Combining (117), (118), (119), (120) with (116) yields

$$\int_{\check{B}_{r_1}} \left|\nabla|u|^2\right|^2 \le C\gamma^2 \int_{\check{B}_r} |\nabla u|^2 + \frac{C}{\gamma^2}\int_{\check{B}_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} + Cr^N\Lambda_0\varepsilon^\beta + C\sqrt{\eta}r^{N-2}. \tag{121}$$

Step 4: Proof of Lemma 4.2 completed.

Recall that

$$4|u|^2\cdot|\nabla u|^2 = 4|u\times\nabla u|^2 + \left|\nabla|u|^2\right|^2,$$

thus, from the Hodge-de Rham decomposition of Step 1,

$$\begin{aligned} 4a_\varepsilon(x)|\nabla u|^2 &= 4|u\times\nabla u|^2 + \left|\nabla|u|^2\right|^2 + 4(a_\varepsilon(x) - |u|^2)|\nabla u|^2 \\ &\le 12\Big[|\nabla\varphi|^2 + |\nabla\xi|^2 + (1-|u|^2)^2|\vec{c}|^2|\log\varepsilon|^2\Big] + \left|\nabla|u|^2\right|^2 + 4(a_\varepsilon(x) - |u|^2)|\nabla u|^2. \end{aligned}$$

Since, by (8),

$$\begin{aligned} \int_{\check{B}_{\delta r}} (1-|u|^2)^2|\vec{c}|^2|\log\varepsilon|^2 &\le \Lambda_0^2\varepsilon^2|\log\varepsilon|^2\int_{\check{B}_{\delta r}}\frac{(1-|u|^2)^2}{\varepsilon^2} \\ &\le C\int_{\check{B}_{\delta r}}\frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} + Cr^N\Lambda_0\varepsilon^\beta \\ &\le \frac{C}{\gamma^2}\int_{\check{B}_r}\frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} + Cr^N\Lambda_0\varepsilon^\beta, \end{aligned}$$

we deduce from (115) in Step 2 and (121) in Step 3, using $4a_\varepsilon(x) \ge 1$ (for $\varepsilon \le \varepsilon_0(\Lambda_0)$ small enough) and (118) (for the last term) that

$$\begin{aligned} E_\varepsilon(\delta r) \le{}& C\Big(\gamma^2 + \delta^N + \frac{\gamma^{-4}}{r^{N-2}}\int_{\check{B}_r}\frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\Big)E_\varepsilon(r) + C\gamma^{-2}\int_{\check{B}_r}\frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \\ &+ Cr^{N-2}\Big(1 + \frac{\gamma^{-4}}{r^{N-2}}\int_{\check{B}_r}\frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\Big)(\eta + \Lambda_0\varepsilon^\beta) + C\sqrt{\eta}r^{N-2}, \end{aligned}$$

which ends the proof. □

Part C: Proof of Theorem 2 completed.

We consider a solution u of (1)-(5) on Ω satisfying

$$\tilde{E}_\varepsilon(u,\bar{r}) \le \eta|\log\varepsilon| \quad \text{and} \quad T_\varepsilon^\nu(r_\varepsilon,\bar{r}) \le \eta|\log\varepsilon|, \tag{122}$$

for a $r_\varepsilon^{1/2} \le \bar{r} \le \min(R, (1+\Lambda_0)^{-1/\nu})$. In Part A, we have exhibited some $r_0 \in (r_\varepsilon, r_\varepsilon^{1/2})$ such that

$$\frac{1}{r_0^{N-2}}\int_{\check{B}_{r_0}}\frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2} \le C(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|, \tag{123}$$

$$\tilde{E}_\varepsilon(r_0) - 2^{N-2}\tilde{E}_\varepsilon(\delta r_0) \le C(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|. \tag{124}$$

We apply Lemma 4.2 to obtain, since $0 < \varepsilon \le r_\varepsilon \le r_0 \le r_\varepsilon^{1/2} \le (1+\Lambda_0|\log\varepsilon|)^{-1}$,

$$\begin{aligned} E_\varepsilon(\delta r_0) \le{}& C\Big(\gamma^2 + \delta^N + \frac{\gamma^{-4}}{r_0^{N-2}}\int_{\check{B}_{r_0}}\frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\Big)E_\varepsilon(r_0) + C\gamma^{-2}\int_{\check{B}_{r_0}}\frac{(a_\varepsilon - |w|^2)^2}{\varepsilon^2} \\ &+ Cr_0^{N-2}\Big(1 + \frac{\gamma^{-4}}{r_0^{N-2}}\int_{\check{B}_{r_0}}\frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\Big)(\eta + \Lambda_0\varepsilon^\beta) + C\sqrt{\eta}r_0^{N-2}. \end{aligned} \tag{125}$$

Therefore, by (123), (124) and dividing (125) by r_0^{N-2},

$$\begin{aligned}
\tilde{E}_\varepsilon(r_0) &\le 2^{N-2}\tilde{E}_\varepsilon(\delta r_0) + C(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta| \\
&\le C\delta^{2-N}\frac{1}{r_0^{N-2}}E_\varepsilon(\delta r_0) + C(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta| \\
&\le C\delta^{2-N}\Big(\gamma^2 + \delta^N + \gamma^{-4}(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|\Big)\tilde{E}_\varepsilon(r_0) \\
&\quad + C\gamma^{-2}(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta| \qquad (126)\\
&\quad + C(1 + \gamma^{-4}(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta|)(\eta + \Lambda_0\varepsilon^\beta) + C\sqrt{\eta}
\end{aligned}$$

We now fix the values of γ and δ. First, we choose δ small enough (depending on N, ν, Ω and the constant C in (12) only) so that

$$C\delta^2 \le 1/4.$$

Next, we fix γ small enough so that

$$C\delta^{2-N}\gamma^2 \le 1/4$$

and thus

$$C\delta^{2-N}(\gamma^2 + \delta^N) \le 1/2.$$

Consequently, for these values of γ and δ, there exist ε_0 and η_0 small such that, for any $\eta \le \eta_0$ and $\varepsilon \le \varepsilon_0$, then

$$C\delta^{2-N}\gamma^{-4}(\eta + \Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1})|\log\delta| \le 1/4.$$

Hence, recalling $r_0 \in (r_\varepsilon, r_\varepsilon^{1/2})$ with $r_\varepsilon = (\varepsilon^\mu|\log\varepsilon|)^{1/(N-1)}$, (126) rewrites, for $0 < \eta < \eta_0$,

$$\tilde{E}_\varepsilon(r_0) \le \frac{3}{4}\tilde{E}_\varepsilon(r_0) + C\sqrt{\eta},$$

provided $0 < \varepsilon \le \varepsilon_0(\Lambda_0, \eta)$ is small enough so that

$$\Lambda_0\varepsilon^\beta \le \sqrt{\eta}.$$

We then infer that for $\eta \le \eta_0$ and $\varepsilon \le \varepsilon_0(\nu, \Lambda_0, N)$,

$$\tilde{E}_\varepsilon(r_0) \le C\sqrt{\eta}.$$

Finally, we apply the monotonicity formula of Lemma 3.4 (note that $\varepsilon \le r_\varepsilon \le (1 + \Lambda_0|\log\varepsilon|)^{-1/\nu}$ for ε small) and obtain, for $\eta \le \eta_0$ and $\varepsilon \le \varepsilon_0(\nu, \Lambda_0, N, \eta)$ since $\Lambda_0\varepsilon^\beta|\log\varepsilon|^{-1} \le \sqrt{\eta}$,

$$\begin{aligned}
\frac{1}{\varepsilon^N}\int_{\Omega\cap B_\varepsilon}(1 - |u|^2)^2 &\le C(\frac{1}{\varepsilon^N}\int_{\Omega\cap B_\varepsilon}(a_\varepsilon - |u|^2)^2 + \Lambda_0^2\varepsilon^\beta) \\
&\le C(\tilde{E}_\varepsilon(\varepsilon) + \sqrt{\eta}) \\
&\le C(\tilde{E}_\varepsilon(r_0) + \sqrt{\eta} + \Lambda_0\varepsilon^2|\log\varepsilon|^3 + \eta) \\
&\le C\sqrt{\eta}.
\end{aligned}$$

We conclude with the following lemma, taken from [BBO] (Lemma III.3 there).

Lemma 4.3. *Assume u satisfies $|\nabla u| \le C/\varepsilon$ in a smooth domain ω and $x_0 \in \bar{\omega}$. Then,*

$$1 - |u(x_0)| \le C(\omega, x_0)\Big(\frac{1}{\varepsilon^N}\int_{\omega\cap B_\varepsilon(x_0)}(1 - |u|^2)^2\Big)^{1/(N+2)}.$$

4.2 Proof of Theorem 3

We only point out the modifications to make for the Neumann case.

Part A: Choosing a "good" radius.

Lemma 4.4. *Assume* $0 < \varepsilon < \delta^{1/(2\beta)}$*, that* u *is a solution of* (1)-(6)*,* $x_0 \in \bar{\Omega}$ *and that*

$$\tilde{E}_\varepsilon(x_0, \bar{r}) \leq \eta |\log \varepsilon|$$

for a $r_\varepsilon^{1/2} \leq \bar{r} \leq \min(R, (1+\Lambda_0)^{-1})$*. Then, there exists a radius* $r_0 \in (r_\varepsilon, r_\varepsilon^{1/2})$ *such that*

- $\displaystyle \frac{1}{r_0^{N-2}} \int_{\check{B}_{r_0}} \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2} \leq C(\eta + \Lambda_0 \varepsilon^\beta |\log \varepsilon|^{-1}) |\log \delta|,$
- $\tilde{E}_\varepsilon(r_0) - 2^{N-2} \tilde{E}_\varepsilon(\delta r_0) \leq C(\eta + \Lambda_0 \varepsilon^\beta |\log \varepsilon|^{-1}) |\log \delta|.$

Proof. The proof is exactely the same as in Lemma 4.1. It suffices to replace (53) by (76). We then proceed as in Lemma 4.1. We bound $\bar{E}_\varepsilon(4r)$ by the monotonicity formula of Proposition 2 and use the fact (as in Lemma 3.14) that the primitive of $\chi_{\{2r \geq d_0\}} \frac{d_0}{4r^2}$ with value 0 in $r = 0$ is uniformly bounded between 0 and 1. $\square$

Part B: δ-energy decay.

Lemma 4.5. *There exist constants* C *and* $\varepsilon_0 > 0$*, depending on* N *and* Ω*, such that, if* u *is a solution of* (1)-(6)*, then*

$$E_\varepsilon(\delta r) \leq C\Big(\gamma^2 + \delta^N + \frac{\gamma^{-4}}{r^{N-2}} \int_{\check{B}_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\Big) E_\varepsilon(r) + C\gamma^{-2}\Big(\int_{\check{B}_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} + \varepsilon^\beta\Big).$$

Proof. For simplicity, we will assume that Ω is locally the half-plane $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}_+$ and that $x_0 = (x_0)_N \vec{e}_N =: ra\vec{e}_N$. First, we consider the rescaled maps on $\check{B}_1(a\vec{e}_N)$, with $\tilde{\varepsilon} := \frac{\varepsilon}{r}$,

$$\hat{u}(x) := u(rx), \qquad \hat{c}(x) := r\frac{|\log \varepsilon|}{|\log \tilde{\varepsilon}|} c(rx) \;\text{ and }\; \hat{d}(x) := r^2 \frac{|\log \varepsilon|^2}{|\log \tilde{\varepsilon}|^2} d(rx).$$

We then define the reflected map $\tilde{u} : \omega := B_1(a\vec{e}_N) \cup B_1(-a\vec{e}_N) \to \mathbb{C}$ with respect to the boundary

$$\tilde{u}(x) := \begin{cases} \hat{u}(x) & \text{if } x \in \check{B}_1(a\vec{e}_N), \\ \hat{u}(x_1, \ldots, x_{N-1}, -x_N) & \text{if } x \in B_1(-a\vec{e}_N) \setminus \check{B}_1(a\vec{e}_N). \end{cases}$$

We extend $\hat{d}$ similarly in $\tilde{d}$ and for $\hat{c}$, we set on $B_1(-a\vec{e}_N) \setminus \check{B}_1(a\vec{e}_N)$

$$\tilde{c}(x) := (\hat{c}_1(x_1, \ldots, x_{N-1}, -x_N), \ldots, \hat{c}_{N-1}(x_1, \ldots, x_{N-1}, -x_N), -\hat{c}_N(x_1, \ldots, x_{N-1}, -x_N)).$$

Since $\vec{c} \cdot n = c_N = 0$ on $\partial \mathbb{R}^N_+$, $\tilde{c}$ is a lipschitz map on ω. Therefore, since $\frac{\partial u}{\partial n} = 0$, $\tilde{u}$ satisfies

$$i|\log \tilde{\varepsilon}| \tilde{c} \cdot \nabla \tilde{u} = \Delta \tilde{u} + \frac{1}{\tilde{\varepsilon}^2} \tilde{u}(\tilde{a}_{\tilde{\varepsilon}} - |\tilde{u}|^2) \qquad \text{in } \omega.$$

In particular, we may apply Lemma A.9 in Appendix A of [BOS] and obtain the desired result. More precisely, we may apply step by step the lines of Lemma A.9 in the Appendix A in [BOS] to obtain, since, with our scalings, $\Lambda_0(\tilde c, \tilde d) \le C$ and the basic estimates of Lemma 2 hold,

$$E_{\tilde\varepsilon}(\tilde u, \omega(\delta)) \le C\Big(\gamma^2 + \delta^N + \gamma^{-4}\int_{\omega(1)} \frac{(\tilde a_{\tilde\varepsilon} - |\tilde u|^2)^2}{\tilde\varepsilon^2}\Big) E_{\tilde\varepsilon}(\tilde u, \omega(1))$$
$$+ C\gamma^{-4}\Big(\int_{\omega(1)} \frac{(\tilde a_{\tilde\varepsilon} - |\tilde u|^2)^2}{\tilde\varepsilon^2} + \varepsilon^\beta\Big), \tag{127}$$

where $\omega(r) := B_r(a\vec e_N) \cup B_r(-a\vec e_N)$ (note that $\omega(1) = \omega$). The only difference with Lemma A.9 in [BOS] is that for our problem, we work with $\omega(r)$ instead of B_r, but this does not affect the arguments. Moreover, by scaling, we have

$$\int_\omega \frac{(\tilde a_{\tilde\varepsilon} - |\tilde u|^2)^2}{\tilde\varepsilon^2} = \frac{1}{2r^{N-2}} \int_{\check B_1(a\vec e_N)} \frac{(\hat a_{\tilde\varepsilon} - |\hat u|^2)^2}{\tilde\varepsilon^2} = \frac{1}{2r^{N-2}} \int_{\check B_r(x_0)} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2},$$

$$E_{\tilde\varepsilon}(\tilde u, \omega) = \frac{1}{2r^{N-2}} E_\varepsilon(u, r) \quad \text{and} \quad E_{\tilde\varepsilon}(\tilde u, \omega(\delta)) = \frac{1}{2r^{N-2}} E_\varepsilon(u, \delta r).$$

Inserting this in (127) yields the conclusion. □

Part C: Proof of Theorem 3 completed.

We consider a solution u of (1)-(6) on Ω satisfying

$$\tilde E_\varepsilon(u, \bar r) \le \eta |\log\varepsilon|, \tag{128}$$

for a $r_\varepsilon^{1/2} \le \bar r \le \min(R, (1+\Lambda_0)^{-1})$. In Part A, we have then exhibited some $r_0 \in (r_\varepsilon, r_\varepsilon^{1/2})$ such that

$$\frac{1}{r_0^{N-2}} \int_{\check B_{r_0}} \frac{(a_\varepsilon - |u|^2)^2}{2\varepsilon^2} \le C(\eta + \Lambda_0 \varepsilon^\beta |\log\varepsilon|^{-1}) |\log\delta|,$$
$$\tilde E_\varepsilon(r_0) - 2^{N-2} \tilde E_\varepsilon(\delta r_0) \le C(\eta + \Lambda_0 \varepsilon^\beta |\log\varepsilon|^{-1}) |\log\delta|.$$

We apply Lemma 4.5 and obtain

$$E_\varepsilon(\delta r_0) \le C\Big(\gamma^2 + \delta^N + \frac{\gamma^{-4}}{r_0^{N-2}} \int_{\check B_{r_0}} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\Big) E_\varepsilon(r_0) + C\gamma^{-2}\Big(\int_{\check B_{r_0}} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} + \varepsilon^\beta\Big).$$

We have therefore the same estimates as in Part C of Appendix A in [BOS] or Part C of the previous subsection 4.1 (see (125), less some terms): the conclusion follows the same lines. □

5 Anchoring condition at the boundary

We prove in this Section that $\tilde{\mathcal V}$ is stationary. The proof of the stationarity inside the domain follows from the curvature equation of [BOS] (Theorem 3 there),

$$H = \star\Big(\vec c \wedge \star \frac{dJ_*}{d\mu_*}\Big) = 0$$

if $\vec c_\varepsilon \to \vec c_0 = 0$ uniformly in $\bar\Omega$.

Notations : We denote $(e_i)_{1\leq i\leq N}$ the canonical basis in $\mathbb{R}^N$, and let $\vec{e}_i := e_i$ in $\bar{\Omega}$, and $\vec{e}_i := \phi^*(e_i)$ in W, so that $(\vec{e}_i)_{1\leq i\leq N}$ is a smooth orthonormal frame in $(\mathcal{M}, g)$, and set $D_i := \frac{\partial}{\partial \vec{e}_i}$, and for a function $v : \mathcal{M} \to \mathbb{R}$, $Dv = (D_i v)_{1\leq i\leq N} \in \mathbb{R}^N = T_v\mathcal{M}$. Let $\tilde{u}_\varepsilon(x) := u_\varepsilon(x)$ in Ω, $\tilde{u}_\varepsilon(x) := u_\varepsilon(\phi(x))$ in W,

$$\nu := \begin{cases} dx & \text{in } \bar{\Omega}, \\ |Jac_x(\phi)|dx & \text{in } W \end{cases}$$

the measure on the riemannian manifold $(\mathcal{M}, g)$, and

$$\tilde{\mu}_\varepsilon := \frac{1}{2|\log\varepsilon|}\Big(\sum_{i=1}^N |D_i\tilde{u}_\varepsilon|^2 + \frac{(1-|\tilde{u}_\varepsilon|^2)^2}{2\varepsilon^2}\Big)d\nu(x)$$

be the energy density measure. We extend $\vec{c}_\varepsilon$, d_ε and $a_\varepsilon = 1 - d_\varepsilon\varepsilon^2|\log\varepsilon|^2$, defined in Ω to Ω_δ by the formulas $\tilde{c}_\varepsilon := \phi^*(\vec{c}_\varepsilon)$, $\tilde{d}_\varepsilon(x) := d_\varepsilon(\phi(x))$ and $\tilde{a}_\varepsilon := a_\varepsilon \circ \phi$ in W. In view of (1) and the Neumann boundary condition (6), $\tilde{u}_\varepsilon$ solves

$$i|\log\varepsilon|\langle\tilde{c}_\varepsilon, d_{\mathcal{M}}\tilde{u}_\varepsilon\rangle = \Delta_{\mathcal{M}}\tilde{u}_\varepsilon + \frac{1}{\varepsilon^2}\tilde{u}_\varepsilon(1-|\tilde{u}_\varepsilon|^2) - |\log\varepsilon|^2\tilde{d}_\varepsilon\tilde{u}_\varepsilon \quad \text{in } (\mathcal{M}, g). \tag{129}$$

Furthermore, it is clear that $E_\varepsilon(\tilde{u}_\varepsilon, \Omega_\delta) \leq CE_\varepsilon(\tilde{u}_\varepsilon, \Omega) \leq CM|\log\varepsilon|$, hence we infer from [JS] and [ABO] that $J\tilde{u}_\varepsilon$ is precompact in $[\mathcal{C}_c^{0,\alpha}(\Omega_\delta)]^*$ for $\alpha \in (0, 1]$ and more precisely, there holds for $\varphi \in \mathcal{C}^{0,1}(\Omega_\delta, \Lambda^2)$ the estimate (40) of Lemma 3.5, namely

$$\Big|\int_{\Omega_\delta}\langle J\tilde{u}_\varepsilon, \varphi\rangle \; d\nu\Big| \leq CM|\varphi|_\infty + CM\varepsilon^\alpha|\log\varepsilon||d\varphi|_\infty. \tag{130}$$

We define

$$\tilde{\alpha}_\varepsilon^{i,j} := \tilde{\mu}_\varepsilon\delta_{i,j} - \frac{(D_i\tilde{u}_\varepsilon, D_j\tilde{u}_\varepsilon)}{|\log\varepsilon|}d\nu(x),$$

so that the matrix $(\tilde{\alpha}_\varepsilon^{i,j})$ is g-symmetric, has trace larger than $(N-2)\tilde{\mu}_\varepsilon$ and eigenvalues less than or equal to $\tilde{\mu}_\varepsilon$. Moreover,

$$|\tilde{\alpha}_\varepsilon^{i,j}| \leq N\tilde{\mu}_\varepsilon. \tag{131}$$

Let also

$$\tilde{\Theta}_*(x) := \liminf_{r\to 0}\frac{\tilde{\mu}_*(B_r(x))}{r^{N-2}}$$

be the $(N-2)$-dimensional density of $\tilde{\mu}_*$, and $\Sigma_{\tilde{\mu}_*} := \{\tilde{\Theta}_* > 0\}$ its geometrical support. It is then clear that $\tilde{\mathcal{V}} := \mathcal{V}(\Sigma_{\tilde{\mu}_*}, \tilde{\Theta}_*)$ is the union of the varifold $\mathcal{V}(\Sigma_{\mu_*}, \Theta_*)$ and its reflection across the boundary.

The argument then follows Appendix B of [BOS]. We fix $\vec{X} \in \mathcal{C}_c^\infty(\Omega_\delta)$ and now, we compute in the riemannian manifold $(\mathcal{M}, g)$, and denote $\cdot$ the scalar product in $\mathcal{M}$. First, we have

$$\int_{\Omega_\delta}\sum_{i=1}^N(\tilde{\alpha}_\varepsilon^{i,j})_{1\leq j\leq N}\cdot D(X\cdot\vec{e}_i)\; d\nu(x) = -\int_{\Omega_\delta}\langle d\tilde{\mu}_\varepsilon, X\rangle - \int_{\Omega_\delta}\sum_{i=1}^N\frac{(D_i\tilde{u}_\varepsilon, D\tilde{u}_\varepsilon)}{|\log\varepsilon|}\cdot D(X\cdot\vec{e}_i). \tag{132}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_{\Omega_\delta}\sum_{i=1}^N[(D_i\tilde{u}_\varepsilon, D\tilde{u}_\varepsilon)]\cdot D(X\cdot\vec{e}_i)\; d\nu(x) &= \sum_{i=1}^N\int_{\Omega_\delta}((D_iD\tilde{u}_\varepsilon; D\tilde{u}_\varepsilon) + (D_i\tilde{u}_\varepsilon, \Delta_{\mathcal{M}}\tilde{u}_\varepsilon)(X\cdot\vec{e}_i)\; d\nu(x)\\ &= -\int_{\Omega_\delta}(\vec{X}\cdot D\tilde{u}_\varepsilon, \Delta_{\mathcal{M}}\tilde{u}_\varepsilon) - \frac{1}{2}\int_{\Omega_\delta}\vec{X}\cdot D(|D\tilde{u}_\varepsilon|^2).\end{aligned}$$

Since $\tilde{u}_\varepsilon$ is a solution of (129), we then infer from (132)

$$\begin{aligned}
&\int_{\Omega_\delta}\sum_{i=1}^{N}(\tilde{\alpha}_\varepsilon^{i,j})_{1\le j\le N}\cdot D(\vec{X}\cdot\vec{e}_i)\\
&\qquad=\frac{1}{|\log\varepsilon|}\int_{\Omega_\delta}\Big(\vec{X}\cdot D\tilde{u}_\varepsilon,\Delta_{\mathcal{M}}\tilde{u}_\varepsilon+\frac{\tilde{u}_\varepsilon}{\varepsilon^2}(\tilde{a}_\varepsilon-|\tilde{u}_\varepsilon|^2)\Big)+\frac{|\log\varepsilon|^2}{2}(\tilde{a}_\varepsilon-|\tilde{u}_\varepsilon|^2)\vec{X}\cdot D\tilde{d}_\varepsilon\\
&\qquad=-\int_{\Omega_\delta}\langle\vec{X},\star(\tilde{c}_\varepsilon\wedge\star J\tilde{u}_\varepsilon)\rangle\,d\nu(x)+\frac{|\log\varepsilon|}{2}\int_{\Omega_\delta}(\tilde{a}_\varepsilon-|\tilde{u}_\varepsilon|^2)\vec{X}\cdot D\tilde{d}_\varepsilon\,d\nu(x)
\end{aligned}\tag{133}$$

Up to a subsequence $\varepsilon_j\to 0$, we may assume that

$$\tilde{\alpha}_\varepsilon^{i,j}\rightharpoonup\tilde{\alpha}_*^{i,j}$$

weakly as measures. Furthermore, we infer from (131) that $|\tilde{\alpha}_*^{i,j}|\le N\tilde{\mu}_*$, so we may write

$$\tilde{\alpha}_*^{i,j}=\tilde{A}^{i,j}\tilde{\mu}_*,$$

for $\tilde{\mu}_*$ almost every $x\in\Omega_\delta$, where $\tilde{A}^{i,j}(x)$ is g-symmetric, with eigenvalues less than or equal to one and trace **equal** to $N-2$ (this follows from Proposition A.1 in Appendix A of [BOS]). We also have

$$\Big|\frac{|\log\varepsilon|}{2}\int_{\Omega_\delta}(\tilde{a}_\varepsilon-|\tilde{u}_\varepsilon|^2)\vec{X}\cdot D\tilde{d}_\varepsilon\,d\nu(x)\Big|\le C(\vec{X})M\Lambda_0\varepsilon|\log\varepsilon|.$$

Since, in the regime of interest for us, we have

$$\tilde{c}_\varepsilon\to 0\qquad\text{uniformly as }\varepsilon\to 0$$

(but $\tilde{c}_\varepsilon|\log\varepsilon|\not\to 0$), we obtain, passing to the limit in (133),

$$\int_{\Omega_\delta}\sum_{i=1}^{N}(\tilde{A}^{i,j}(x))_{1\le j\le N}\cdot D(\vec{X}\cdot\vec{e}_i)d\tilde{\mu}_*\llcorner\Sigma_{\tilde{\mu}_*}=0.\tag{134}$$

To be very precise, the convergence towards zero for the first term in (133) is deduced from (130) since $\tilde{c}_\varepsilon\to 0$, thus $\varphi:=\vec{X}\wedge\tilde{c}_\varepsilon\to 0$ uniformly, with a gradient uniformly bounded. Since $\vec{X}$ is arbitrary in $\mathcal{C}_c^\infty(\Omega_\delta)$, this states that the reflected varifold $\tilde{\mathcal{V}}=\delta_{\tilde{A}(x)}\tilde{\mu}_*\llcorner\Sigma_{\tilde{\mu}_*}(x)$ is stationary (see [S]) in $(\mathcal{M},g)$, which concludes the proof of Theorem 1. In the case $\vec{c}_\varepsilon\to\vec{c}_0\neq 0$, equation (134) becomes

$$\int_{\Omega_\delta}\sum_{i=1}^{N}(\tilde{A}^{i,j}(x))_{1\le j\le N}\cdot D(\vec{X}\cdot\vec{e}_i)d\tilde{\mu}_*\llcorner\Sigma_{\tilde{\mu}_*}=-\int_{\Omega_\delta}\langle\star(\tilde{c}_0\wedge\star\frac{d\tilde{J}_*}{d\tilde{\mu}_*}),\vec{X}\rangle\,d\nu$$

and this completes the proof of (11). □

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